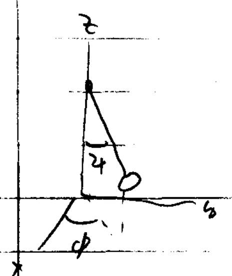
Lecture 13

Last time, talked about non inertial effects on Earth

Working on Foucault pendulum

= 3D pendulum, in rotating frame, colatitude θ



Using Newton, got

$$m\ddot{z} = -mg + T\cos\psi + 2m\omega\dot{y}\sin\theta$$

$$m\ddot{x} = -T\sin\psi\cos\phi + 2m\omega\dot{y}\cos\theta$$

$$m\ddot{y} = -T\sin\psi\sin\phi \underbrace{-2m\omega(\dot{x}\cos\theta + \dot{z}\sin\theta)}_{\text{Coriolis}}$$

In limit of small oscillations, $T \approx mg$ and $\dot{z} \approx 0$

Use $x = l\sin\psi\cos\phi$

$y = l\sin\psi\sin\phi$

$$\ddot{x} = -\frac{g}{l}x + 2\omega\dot{y}\cos\theta$$

Define $\Omega^2 = \frac{g}{l}$, $\omega_\perp = \omega\cos\theta$

$$\ddot{y} = -\Omega^2 y - 2\omega_\perp\dot{x}$$

Oscillation coupled.

Nice way to solve: $\xi \equiv x + iy$

$$\ddot{\xi} = \ddot{x} + i\ddot{y} = -\Omega^2 x + 2\omega_\perp\dot{y} - \Omega^2 y - 2\omega_\perp\dot{x}$$

$$= -\Omega^2\xi - 2i\omega_\perp\dot{\xi}$$

So $\ddot{\xi} + 2i\omega_\perp\dot{\xi} + \Omega^2\xi = 0$

Try $\rho = e^{i\sigma t}$

$$-\sigma^2 - 2\omega_\perp \sigma + \Omega^2 = 0$$

$$\sigma = -\tfrac{1}{2}\left[2\omega_\perp \pm \sqrt{4\omega_\perp^2 + 4\Omega^2}\right]$$

$$= -\omega_\perp \pm \sqrt{\omega_\perp^2 + \Omega^2}$$

$\omega_\perp < \omega_e \leq 10^{-4}\,s^{-1}$. If $l = 20\,m$, $\Omega = \sqrt{\frac{g}{l}} = 1\,s^{-1}$

So $\omega_\perp \ll \Omega$: expand

$$\sigma = -\omega_\perp \pm \Omega\Big(1 + \underbrace{\cancel{\frac{\omega_\perp^2}{2\Omega^2}}}_{2^{nd}\text{ order}}\Big)$$

$$\sigma = -\omega_\perp \pm \Omega$$

$$\rho(t) = A e^{i\sigma_+ t} + B e^{i\sigma_- t}$$

$$= e^{-i\omega_\perp t}\left(A e^{i\Omega t} + B e^{-i\Omega t}\right)$$

Say initial conditions $x(0) = a$ $y(0) = \dot{x} = \dot{y} = 0$

$$\rho(0) = a = A + B$$

$$\dot{\rho}(0) = 0 = -i(\omega_\perp + \Omega)A - i(\omega_\perp - \Omega)B$$

$$= \underbrace{\omega_\perp (A+B)}_{\text{neglect}} + \Omega(A-B)$$

$$\Rightarrow A = B = \frac{a}{2}$$

$$\rho(t) = a e^{-i\omega_\perp t} \cos \Omega t$$

$x(t) = a \cos \omega_\perp t \cos \Omega t$ $y(t) = -a \sin \omega_\perp t \cos \Omega t$

Energy transferred between x + y, as usual

Plane of oscillation: $\tan\phi = \frac{y}{x} = -\tan\omega_\perp t$

$$\phi = -\omega_\perp t$$

Plane precesses at rate $\omega_\perp = \omega_e \cos\theta$

Measures absolute rotation of earth

So, we did that from Newtonian point of view
Wouldn't it be nicer with Lagrangians?
Yes!

Actual pretty straightforward:
use body coord $\vec{r}$ as generalized coord in L

Then eqn of motion is $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i}$
as usual

First, prove a useful fact

If $L_A = L_B + \frac{d}{dt} f(q, \dot{q}, t)$

Then L_A + L_B generate same eqns of motion

Prove with Hamiltons principle

$$S_A = \int_{t_1}^{t_2} L_A \, dt$$

$$= \int L_B + \frac{df}{dt} \, dt = \int L_B \, dt + f(t_2) - f(t_1)$$

$S_A = S_B + \text{const}$

$\uparrow$ endpoints don't vary

So $\delta S_A = \delta S_B \Rightarrow$ same motion

So, let $L_0 =$ Lagrangian in inertial frame $= \frac{1}{2} m v_0^2 - V_0(\vec{r}_0)$

First handle translation:

If origin of body frame at $\vec{R} = \vec{R}(t)$

$$\vec{r}_0 = \vec{r} + \vec{R}$$

Just translate coords in V: $V(\vec{r}) = V_0(\vec{r}_0)$

Value of potential at particle location doesn't change

For T, use $\dot{\vec{r}}_0 = \dot{\vec{r}} + \dot{\vec{R}}$

$$T = \frac{1}{2} m (\vec{v} + \dot{\vec{R}})^2 = \frac{1}{2} m v^2 + m \vec{v} \cdot \dot{\vec{R}} + \frac{1}{2} m \dot{R}^2$$

$\downarrow$

specified function of t

$= \frac{d}{dt}\left[\int \dot{R}^2 dt\right]$

Can drop

Also, $m\vec{v} \cdot \dot{\vec{R}} = \underbrace{\frac{d}{dt}\left[m \dot{\vec{R}} \cdot \vec{r}\right]}_{\text{drop}} - m\vec{r} \cdot \ddot{\vec{R}}$

So $T \to \frac{1}{2} m v^2 - m\vec{r} \cdot \ddot{\vec{R}}$

$$\boxed{L = \frac{1}{2} m v^2 - m\vec{r} \cdot \ddot{\vec{R}} - V(\vec{r})}$$

Looks like extra potential $U_{react} = m\vec{r}\cdot\ddot{\vec{R}}$

Gives reaction force

$$\vec{F} = -\vec{\nabla} U_{react} = -m\ddot{\vec{R}}$$

Now put in rotation:

Again, in potential term, just rewrite in terms of new coord

- Same for U_{react}

In T, use $\vec{v}_0 = \vec{v} + \vec{\omega}\times\vec{r}$

$$\frac{1}{2}mv_0^2 = \frac{1}{2}m\,|\vec{v} + \vec{\omega}\times\vec{r}|^2$$

$$= \frac{1}{2}m\left[v^2 + 2\vec{v}\cdot(\vec{\omega}\times\vec{r}) + |\vec{\omega}\times\vec{r}|^2\right]$$

So all together

$$\boxed{L = \frac{1}{2}mv^2 + m\vec{v}\cdot(\vec{\omega}\times\vec{r}) + \frac{1}{2}m|\vec{\omega}\times\vec{r}|^2 - m\vec{r}\cdot\ddot{\vec{R}} - V(\vec{r})}$$

Check equation of motion

$$\frac{\partial L}{\partial v_i} = mv_i + m(\vec{\omega}\times\vec{r})_i$$

$$\frac{\partial L}{\partial x_i} = \underbrace{m\vec{v}\cdot\left(\vec{\omega}\times\frac{\partial\vec{r}}{\partial x_i}\right)}_{=m\frac{\partial\vec{r}}{\partial x_i}\cdot(\vec{v}\times\vec{\omega})} + \underbrace{m(\vec{\omega}\times\vec{r})\cdot\left(\vec{\omega}\times\frac{\partial\vec{r}}{\partial x_i}\right)}_{=m\frac{\partial\vec{r}}{\partial x_i}\cdot\left[(\vec{\omega}\times\vec{r})\times\vec{\omega}\right]} - m\ddot{R}_i - \frac{\partial V}{\partial x_i}$$

$$= m(\vec{v}\times\vec{\omega})_i$$

using $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

and $\frac{d}{dt}\frac{\partial L}{\partial \dot{v}_i} = m\dot{v}_i + m\left(\frac{d\vec{\omega}}{dt}\times\vec{r}\right)_i + m(\vec{\omega}\times\vec{v})_i$

$$= m(\vec{v}\times\vec{\omega})_i + m\left[(\vec{\omega}\times\vec{r})\times\vec{\omega}\right]_i$$

$$- m\ddot{R}_i - \frac{\partial V}{\partial x_i}$$

Write as vector:

$$\boxed{m\dot{\vec{v}} = -2m\,\vec{\omega}\times\vec{v} - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r}) - m\,\frac{d\vec{\omega}}{dt}\times\vec{r} - m\ddot{\vec{R}} - \vec{\nabla}V}$$

Correct ✓

Interpretation

$m\vec{r}\cdot\ddot{\vec{R}}$: looks like a potential

$-\frac{1}{2}m|\vec{\omega}\times\vec{r}|^2 = -\frac{1}{2}m\omega^2r^2\sin^2\Theta$

looks like potential pushing particle away from origin

$m\vec{v}\cdot(\vec{\omega}\times\vec{r})$: Gives Coriolis & acceleration term
Has $\vec{v}$, not like a potential
Better to think of as part of T

So effective potential $V_{eff} = V(\vec{r}) + m\vec{r}\cdot\ddot{\vec{R}} - \frac{1}{2}m|\vec{\omega}\times\vec{r}|^2$

$$L = \frac{1}{2}mv^2 + m\vec{v}\cdot(\vec{\omega}\times\vec{r}) - V_{eff}$$

Let's calculate Hamiltonian

$H = \vec{p} \cdot \vec{v} - L$

$p_i = \frac{\partial L}{\partial v_i} = m v_i + m(\vec{\omega} \times \vec{r})_i$

$\vec{p} = m(\vec{v} + \vec{\omega} \times \vec{r}) \neq m\vec{v}$!

$H = m v^2 + m \vec{v} \cdot \vec{\omega} \times \vec{r} - L$

$$\boxed{H = \frac{1}{2} m v^2 + V_{eff}(\vec{r})}$$

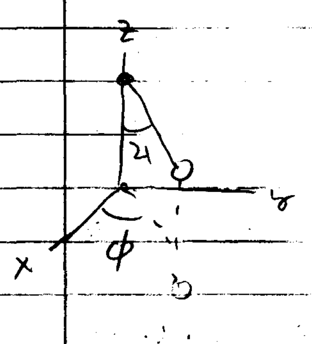
Coriolis term absent!
(Never does work)

Note $H \neq E$ here, since
transformation is time dependent

But if $\vec{\omega}, \dot{\vec{R}} =$ constants
$H =$ conserved

Acts like energy in rotating frame

See how this applies to Foucault problem



$\vec{\omega} = \omega(\cos\theta \hat{z} - \sin\theta \hat{x})$

$V_{eff} = mgz$

$\Rightarrow$ effects of Earth rotation included in g

So $L = \frac{1}{2} m v^2 + m\vec{v} \cdot (\vec{\omega} \times \vec{r}) - mgz$

Use ϕ and ψ as coordinates

$v^2 = \ell^2 \dot{\psi}^2 + \ell^2 \sin^2\psi \, \dot{\phi}^2$

$z = -l\cos\psi$

Cross product term:

$$m\vec{v}\cdot(\vec{\omega}\times\vec{r}) = m\vec{\omega}\cdot(\vec{r}\times\vec{v})$$

$$= \vec{\omega}\cdot\vec{L} \quad \text{angular momentum } m\vec{r}\times\vec{v}$$

Know $L_z = ml^2\sin^2\psi\,\dot{\phi}$

$$L_x = m(y\dot{z} - z\dot{y})$$

$$y = l\sin\psi\cos\phi$$
$$\dot{y} = l[\dot{\psi}\cos\psi\cos\phi - \dot{\phi}\sin\psi\sin\phi]$$

$$z = -l\cos\psi$$
$$\dot{z} = l\dot{\psi}\sin\psi$$

$$L_x = ml^2[\sin\psi\cos\phi\sin\psi\,\dot{\psi} + \cos\psi(\dot{\psi}\cos\psi\cos\phi - \dot{\phi}\sin\psi\sin\phi)]$$

$$= ml^2[\dot{\psi}(\sin^2\psi + \cos^2\psi)\cos\phi - \dot{\phi}\cos\psi\sin\psi\sin\phi]$$

$$= ml^2[\dot{\psi}\cos\phi - \dot{\phi}\cos\psi\sin\psi\sin\phi]$$

So $\mathcal{L} = \frac{1}{2}ml^2(\dot{\psi}^2 + \sin^2\psi\,\dot{\phi}^2) + mgl\cos\psi$

$$+ ml^2\omega[\dot{\phi}\cos\Theta\sin^2\psi - \dot{\psi}\sin\Theta\cos\phi + \dot{\phi}\sin\Theta\cos\psi\sin\psi\sin\phi]$$

Look at eqn for ϕ

$L =$

$$\frac{\partial L}{\partial \dot\phi} = ml^2 \sin^2\psi\, \dot\phi + ml^2\omega\left[\cos\theta \sin^2\psi + \sin\theta\cos\psi\sin\psi\sin\phi\right]$$

$$\frac{d}{dt}[\;] = ml^2\frac{d}{dt}\left[\sin^2\psi\,(\dot\phi + \omega_\perp)\right]$$

$$+ml^2\omega\sin\theta\left[\dot\psi\,(-\sin^2\psi+\cos^2\psi)\sin\phi + \dot\phi\cos\psi\sin\psi\cos\phi\right]$$

and

$$\frac{\partial L}{\partial \phi} = ml^2\omega\left[\dot\psi\sin\theta\sin\phi + \dot\phi\sin\theta\cos\psi\sin\psi\cos\phi\right]$$

So

$$\frac{d}{dt}\left[\sin^2\psi\,(\dot\phi+\omega_\perp)\right] + \omega\sin\theta\,\dot\psi\,(\cos^2\psi - \sin^2\psi)\sin\phi$$

$$+\omega\sin\theta\,\dot\phi\,\cancel{\cos\psi\sin\psi\cos\phi} - \omega\sin\theta\,\dot\psi\sin\phi\,(\cos^2\psi+\sin^2\psi)$$

$$-\omega\sin\theta\,\dot\phi\,\cancel{\cos\psi\sin\psi\cos\phi} = 0$$

$$\underbrace{\frac{d}{dt}[\quad]}_{\substack{2^{nd}\text{ order}\\ \text{in }\psi}} \underbrace{- 2\omega\sin\theta\sin\phi\,\dot\psi\sin^2\psi}_{\substack{3^{rd}\text{ order}\\ \text{in }\psi \to \text{ drop for}\\ \text{small oscillations}}} = 0$$

So $\sin^2\psi\,(\dot\phi+\omega_\perp) = \text{constant}$

If $\psi = 0$ at any point in trajectory, need constant $= 0$

Then $\boxed{\dot\phi = -\omega_\perp}$ as before