# Lecture 12

Last time, started discussing non-inertial frames

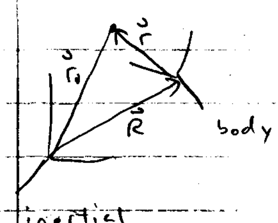
Force law in rotating frame:

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{body} = \underbrace{\vec{F}}_{\text{real forces}} - \underbrace{2m\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{body}}_{\text{Coriolis}} - \underbrace{m\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{\text{Centrifugal}} - \underbrace{m\frac{d\vec{\omega}}{dt}\times\vec{r}}_{\text{tangential acceleration}}$$

$\vec{\omega} = \vec{\omega}(t) =$ angular velocity of body frame w/respect to inertial frame

In general, body frame could also be accelerating

Say origin at $\vec{R}(t)$ relative to inertial frame




So $\vec{r}_0 = \vec{r} + \vec{R}$

$$\left(\frac{d^2\vec{r}_0}{dt^2}\right)_{inert} = \left(\frac{d^2 r}{dt^2}\right)_{inert} + \left(\frac{d^2\vec{R}}{dt^2}\right)_{inert}$$

or

$$\left(\frac{d^2\vec{r}_0}{dt^2}\right)_{inert} = \left(\frac{d^2 R}{dt^2}\right)_{inert} + \left(\frac{d^2 r}{dt^2}\right)_{body} + 2\vec{\omega}\times\vec{v} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \frac{d\vec{\omega}}{dt}\times\vec{r}$$

$$= \frac{1}{m}\vec{F}$$

So

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{body} = F - 2m\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{body} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - m\frac{d\vec{\omega}}{dt}\times\vec{r} - m\left(\frac{d^2\vec{R}}{dt^2}\right)_{inert}$$

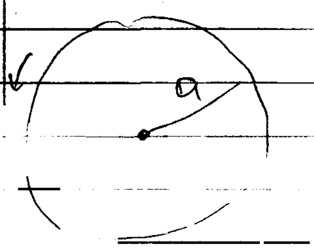
$\frac{d^2R}{dt^2}$ term is usual reaction force ... push you back in seat when car accelerates

Note: book uses $\vec{a}$ instead of $\vec{R}$

Not good... easy to think $\vec{a}$ = acceleration
Don't get confused

Note that there is interplay between different terms, depending on where you put origin.

For instance: merry-go-round with constant $\vec{\omega}$




Could put origin at center
Centrifugal force $F_c = m\omega^2 \rho \, \hat{\rho}$

At edge, force $m\omega^2 a$ away from center

But could instead put origin at edge.
Then $F_c = 0$ since $\rho$ = distance from origin = 0

But now origin is accelerating

$$\vec{R} = \hat{x}^0 a \cos\omega t + \hat{y}^0 a \sin\omega t$$

$\hat{x}^0, \hat{y}^0$ = inertial unit vectors

$$\ddot{\vec{R}} = -\omega^2 \vec{R}$$

Reaction force $F_R = +m\omega^2 \vec{R}$

magnitude $m\omega^2 a$, directed away from center

Here reaction & centrifugal forces add to same thing, no matter where origin is.

Explore in HW

Most common non-inertial frame = Earth

Rotation period $\tau_e = 23.93$ hours
(relative to stars)

$$\omega_e = \frac{2\pi}{\tau_e} = 7.3\times 10^{-5}\ s^{-1}$$
fairly constant

Direction of $\vec{\omega}$ = out of north pole

Earth orbits sun, sun orbits galaxy, ...
also contribute non-inertial effects
but much smaller, we'll ignore

Set up body frame
Origin at Earth's center
z-axis along north pole

Then $m\ddot{\vec{r}} = \vec{F}_g + \vec{F}' - 2m\vec{\omega}\times\dot{\vec{r}} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$

$$\vec{F}_g = \text{gravity force} = -\frac{GM_e m\vec{r}}{r^3}$$

$\vec{F}'$ = other forces

Simplest problem: put particle on scale

What does it weigh?

Particle stationary, so $\dot{\vec{r}}, \ddot{\vec{r}} = 0$

$$0 = \vec{F}_g + \vec{F}_{scale} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

So $\quad \vec{F}_{scale} = -\vec{F}_g + m\vec{\omega}\times(\vec{\omega}\times\vec{r})$

Scale reads $|\vec{F}_{scale}|$, plumb bob gives direction

By definition, scale reads weight $mg$

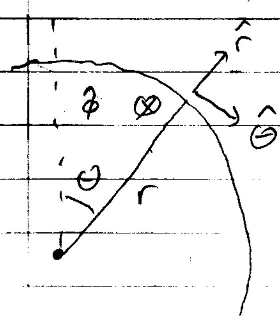
plumb bob hangs in direction $\hat{g}$

So define $\quad \vec{g} = -\dfrac{GM_e}{r^3}\,\vec{r} - \vec{\omega}\times(\vec{\omega}\times\vec{r})$

No way to distinguish gravity and centrifugal force

Since Earth $\simeq$ sphere, useful in spherical coords

Spherical unit vectors




$\hat{r}$ = direction of increasing $r$

$\hat{\theta}$ = " " $\theta$

$\hat{\phi}$ = " " $\phi$

Then $\quad \hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$

$$\vec{\omega} = \omega\hat{z} = \omega(\hat{r}\cos\theta - \hat{\theta}\sin\theta)$$

$$\vec{\omega}\times\vec{r} = \omega r\left[\hat{r}\times\hat{r}\cos\theta - \hat{r}\times\hat{\theta}\sin\theta\right]$$

$$= \omega r\sin\theta\ \hat{\phi}$$

and $\quad \vec{\omega}\times(\vec{\omega}\times\vec{r}) = \omega^2 r\sin\theta\left[\hat{r}\times\hat{\phi}\cos\theta - \hat{\theta}\times\hat{\phi}\sin\theta\right]$

$$= \omega^2 r\sin\theta\left[-\hat{\theta}\cos\theta - \hat{r}\sin\theta\right]$$

So $\vec{g} = -\dfrac{GM\hat{r}}{r^3} - \vec{\omega}\times(\vec{\omega}\times\vec{r})$

$$= \left[-\frac{GM}{r^2} + \omega^2 r\sin\theta\right]\hat{r} + \omega^2 r\sin\theta\cos\theta\,\hat{\theta}$$

To good approximation, $r = R_e$ = radius of Earth
= 6380 km

Compare numbers: $\dfrac{GM}{R_e^2} = 9.8\ ^{m}/_{s^2}$

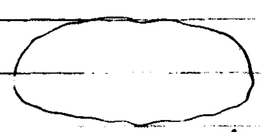
$\omega^2 R_e = 3.4\ ^{cm}/_{s^2}$

Small correction, but measurable

But really more complicated:
Earth ≠ sphere

On geologic time scale, Earth ≈ fluid
Flows in response to force

$\hat{\theta}$ term in $\vec{g}$ drives mass towards equator



exaggerated

So actually $r$ is different at poles and equator

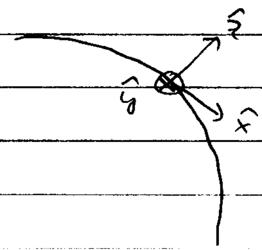
Get to work out in HW

Main points: "sea level" defines equipotential surface
Gradient of potential gives $\vec{g}$
Find $g_{pole} - g_{equator} \approx 0.5\%$

Earth-centered coords not always best

Sometimes better to have $z =$ (local) "up"

Define local coords




$\hat{z} = \hat{r}$

$\hat{x} = \hat{\theta}$

$\hat{y} = \hat{\phi}$

(ignoring ellipticity of earth)

$\theta =$ "co-lattitude"

So $\vec{\omega} = \omega(\cos\theta\,\hat{z} - \sin\theta\,\hat{x})$ here

Consider harder problem: falling object

$\vec{v} \neq 0$, get Coriolis force

Eqn of motion: $m\dot{\vec{v}} = m\vec{g} - 2m\,\vec{\omega}\times\vec{v}$

$$\boxed{\dot{\vec{v}} = \vec{g} - 2\vec{\omega}\times\vec{v}}$$

$$= -g\hat{z} - 2\omega\left[\cos\theta\;\hat{z}\times\vec{v} - \sin\theta\,\hat{x}\times\vec{v}\right]$$

Can't solve analyticlly

Use approximate method

valid if $g \gg \omega v$

$v \ll g/\omega \approx 10^5$ m/s

Perturbative approach

Write $\vec{r} = \vec{r}_0(t) + \vec{r}_1(t)$

$\vec{r}_0 =$ solution for $\omega = 0$

$\vec{r}_1 =$ correction, $1^{st}$ order in $\omega$

Put in eqn of motion $\ddot{\vec{r}} = g - 2\vec{\omega}\times\dot{\vec{r}}$

$$\ddot{\vec{r}}_0 + \ddot{\vec{r}}_1 = \vec{g} - 2\vec{\omega}\times\dot{\vec{r}}_0 - 2\vec{\omega}\times\dot{\vec{r}}_1$$

Order in $\omega$: 0 1 0 1 2

Discard $2^{nd}$ order term, leaves

$$\ddot{\vec{r}}_0 = g \qquad \ddot{\vec{r}}_1 = -2\vec{\omega}\times\dot{\vec{r}}_0$$

Solve for $\vec{r}_0(t)$ first, plug in to get $\vec{r}_1$

Example, $\vec{r}_0(0) = h\hat{z}$ $\qquad \vec{v}(0) = 0$

Then $\vec{r}_0(t) = \left(h - \frac{1}{2}gt^2\right)\hat{z}$ simple falling

$$\dot{\vec{r}}_0 = -gt\hat{z}$$

So $\ddot{\vec{r}}_1 = -2\vec{\omega}\times(-gt\hat{z}) = 2gt\,\vec{\omega}\times\hat{z}$

$$\vec{\omega} = \omega(\cos\theta\,\hat{z} - \sin\theta\,\hat{x})$$

$$\vec{\omega}\times\hat{z} = \omega\sin\theta\,\hat{y}$$

$$\ddot{\vec{r}}_1 = 2g\omega t\sin\theta\,\hat{y}$$

$$\boxed{\vec{r}_1(t) = \frac{1}{3}g\omega t^3\sin\theta\,\hat{y}}$$ horizontal deviation

Some numbers: $h = 100\,m$

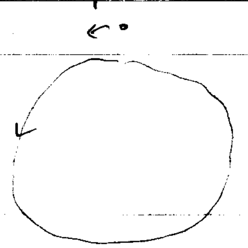
$\theta = 58°$ (C'ville)

$$t = \sqrt{\frac{2h}{g}} = 4.5\,s$$

Get displacement $y_1 = 1.8$ cm

Falling particle moves towards east (= direction of rotation)

Interpret in inertial frame:




Elevated particle moving east faster than surface is

Gets ahead of surface as it falls

Use perturbative technique for any motion:

Solve for $\vec{r}_0(t)$ via freshman physics

Integrate $\ddot{\vec{r}}_1 = -2\vec{\omega}\times\dot{\vec{r}}_0$ to get correction

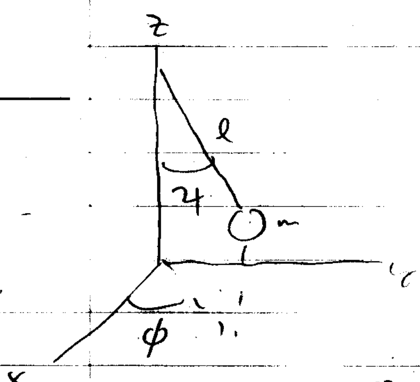
Last example: Foucault pendulum

= 3D pendulum on Earth

Perturbative method fails as $t \to \infty$

correction no longer small

Can solve exactly (in small oscillation limit)




Mass obeys

$$m\ddot{\vec{r}} = m\vec{g} + \vec{T} - 2m\vec{\omega}\times\dot{\vec{r}}$$

$\vec{T}$ = tension

$$\vec{T} = T(-\sin\psi\cos\phi\,\hat{x} - \sin\psi\sin\phi\,\hat{y} + \cos\psi\,\hat{z})$$

Also need $\vec{\omega}\times\dot{\vec{r}}$

$$= \omega(\cos\theta\,\hat{z} - \sin\theta\,\hat{x})\times(\dot{x}\,\hat{x} + \dot{y}\,\hat{y} + \dot{z}\,\hat{z})$$

$$= \omega\left[-\dot{y}\cos\Theta\,\hat{x} + (\dot{x}\cos\Theta + \dot{z}\sin\Theta)\hat{y} - \dot{y}\sin\Theta\,\hat{z}\right]$$

Consider each component

$\hat{z}$: $m\ddot{z} = -mg + T\cos\psi + 2m\omega\dot{y}\sin\Theta$

Note $\omega\dot{y} \ll g$, so neglect this term
Get regular pendulum equation

Small oscillation $\psi \approx 0$, $T \approx mg$, $\ddot{z} \approx 0$

For $x$ & $y$:

$$m\ddot{x} = -T\sin\psi\cos\phi + 2m\omega\dot{y}\cos\Theta$$

$$m\ddot{y} = -T\sin\psi\sin\phi - 2m\omega\left(\dot{x}\cos\Theta + \cancel{\dot{z}\sin\Theta}\right)$$

($z \approx$ const)

Here $\omega\dot{y}$ is small, but so is $T\sin\psi$
keep both terms

Use $T = mg$
$x = l\sin\psi\cos\phi$
$y = l\sin\psi\sin\phi$

$$\boxed{\begin{aligned}\ddot{x} &= -\Omega^2 x + 2\omega_\perp \dot{y}\\ \ddot{y} &= -\Omega^2 y - 2\omega_\perp \dot{x}\end{aligned}} \qquad \boxed{\begin{aligned}\omega_\perp &= \omega\cos\Theta\\ \Omega^2 &= \frac{g}{l}\end{aligned}}$$

Coupled oscillators, but velocity-dependent coupling

Nice way to solve: write $\zeta = x + iy$

$$\ddot{\xi} = \ddot{x} + i\ddot{y} = (-\Omega^2 x + 2\omega_\perp \dot{y}) + i(-\Omega^2 y - 2\omega_\perp \dot{x})$$

$$= -\Omega^2 (x + iy) - 2i\omega_\perp (\dot{x} + i\dot{y})$$

$$= -\Omega^2 \xi - 2i\omega_\perp \dot{\xi}$$

So $\ddot{\xi} + 2i\omega_\perp \dot{\xi} + \Omega^2 \xi = 0$

Look for solution $\xi(t) = A e^{-i\sigma t}$

$$-\sigma^2 + 2\omega_\perp \sigma + \Omega^2 = 0$$

$$\sigma = -\frac{1}{2}\left(-2\omega_\perp \pm \sqrt{4\omega_\perp^2 + 4\Omega^2}\right)$$

$$\boxed{\sigma = \omega_\perp \pm \sqrt{\omega_\perp^2 + \Omega^2}}$$

But $\omega_\perp < \omega < 10^{-4}\ s^{-1}$

If $l = 20$ m, $\Omega = \sqrt{\frac{g}{l}} \approx 1\ s^{-1}$

Can assume $\omega_\perp \ll \Omega$

$$\sqrt{\Omega^2 + \omega_\perp^2} \to \Omega + \frac{\omega_\perp^2}{2\Omega} \approx \Omega$$

$$\sigma_\pm = \omega_\perp \pm \Omega$$

$$\xi(t) = A e^{-i\sigma_+ t} + B e^{-i\sigma_- t}$$

$$= e^{-i\omega_\perp t}\left(A e^{-i\Omega t} + B e^{i\Omega t}\right)$$

Say initial conditions $x(0) = a$, $y(0) = 0$

$\dot{x} = \dot{y} = 0$

So $\varsigma(0) = a = A + B$

$$\dot{\varsigma}(0) = 0 = -i(\omega_\perp + \Omega)A - i(\omega_\perp - \Omega)B$$

$$= \underbrace{\omega_\perp (A+B)}_{\text{neglect}} + \Omega(A - B)$$

$$A \approx B = \frac{a}{2}$$

$$\varsigma(t) = \frac{a}{2} e^{-i\omega_\perp t} \left(e^{-i\Omega t} + e^{i\Omega t}\right)$$

$$= a e^{-i\omega_\perp t} \cos \Omega t$$

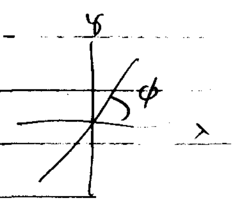
Finally, $x(t) = \operatorname{Re} \varsigma$

$$= a \cos \omega_\perp t \cos \Omega t$$

$$y(t) = \operatorname{Im} \varsigma$$

$$= -a \sin \omega_\perp t \cos \Omega t$$

Mass starts oscillating along $x$ (N-S)
then rotates to $y$ (E-W)




Plane of oscillation

$$\tan\phi = \frac{y}{x} = \tan \omega_\perp t$$

$$\phi = \omega_\perp t$$

So plane of oscillation precesses at $\omega_\perp = \omega \cos\theta$

Measures absolute rotation of Earth