Lecture 11

Midterm exam: Scheduled for 10/9
Delay to 10/16?

Finished discussion of Lagrangians

Continue to use when convenient

Turn to Ch. 2: Non-inertial frames

When useful:

Problems involving "platform" rotating at pre-determined rate $\omega(t)$

Often easier to solve in co-rotating frame

Obvious example: Earth

Rotates at $\omega_e = 7.3 \times 10^{-5}$ rad/s

so "lab frame" is really non-inertial

Clues that you should use non-inertial frame:

- Latitude on earth is specified
- Motion takes place on a platform (or hoop, or box, etc) that has specified acceleration or rotation

Clues that inertial frame better

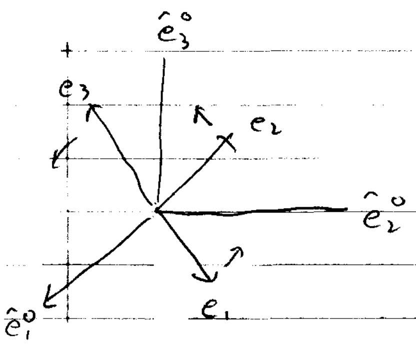
- Dynamics of accelerating/rotating body are of interest (ie spinning top, rocket)
- Conservation of momentum or angular momentum is important

Rotating Frame

Two Cartesian coord systems

Inertial frame axes $\hat{e}_1^o, \hat{e}_2^o, \hat{e}_3^o$

Rotating frame axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$ "body frame"



Common origin

Relative to inertial frame,

$$\hat{e}_i = \hat{e}_i(t)$$

Consider vector $\vec{A}$

$$\vec{A} = \sum_i A_i^o \hat{e}_i^o = \sum_i A_i \hat{e}_i$$

Important: can write "$\vec{A}$" w/o specifying frame

ie, particle has definite position indep of coords

But, this not true for $\frac{d\vec{A}}{dt}$

Particle might appear to be stationary in one frame, moving in another

Need to specify frame of derivative:

$$\left(\frac{d\vec{A}}{dt}\right)_{inert} \quad or \quad \left(\frac{d\vec{A}}{dt}\right)_{body}$$

$$\left(\frac{d\vec{A}}{dt}\right)_{inert} = \sum_i \frac{dA_i^o}{dt} \hat{e}_i^o$$

$$\left(\frac{d\vec{A}}{dt}\right)_{body} = \sum_i \frac{dA_i}{dt} \hat{e}_i$$

Also have $\left(\frac{d\vec{A}}{dt}\right)_{inert} = \frac{d}{dt} \sum_i A_i \hat{e}_i(t)$

$$= \sum_i \frac{dA_i}{dt} \hat{e}_i + A_i \frac{d\hat{e}_i}{dt}$$

So $\left(\frac{d\mathring{A}}{dt}\right)_{inert} = \left(\frac{d\mathring{A}}{dt}\right)_{body} + \sum_i A_i \frac{d\hat{e}_i}{dt}$

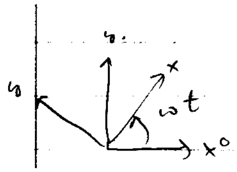
really $\left(\frac{d\hat{e}_i}{dt}\right)_{inert}$

but $\left(\frac{de_i}{dt}\right)_{body} \equiv 0$

Example:

inertial frame $(x^\circ, y^\circ, z^\circ)$

rotating frame (x, y, z)



$$\hat{e}_x = \hat{e}^\circ_x \cos\omega t + \hat{e}^\circ_y \sin\omega t$$
$$\hat{e}_y = -\hat{e}^\circ_x \sin\omega t + \hat{e}^\circ_y \cos\omega t$$

Say $\vec{A} = \cos\Omega t \; \hat{e}_x + \sin\Omega t \; \hat{e}_y$

Pretty obvious $\vec{A} = \cos(\Omega+\omega)t \; \hat{e}^\circ_x + \sin(\Omega+\omega)t \; \hat{e}^\circ_y$

$$\left(\frac{d\vec{A}}{dt}\right)_{inert} = (\Omega+\omega)\left[-\sin(\Omega+\omega)t \; \hat{e}^\circ_x + \cos(\Omega+\omega)t \; \hat{e}^\circ_y\right]$$

and $\left(\frac{d\vec{A}}{dt}\right)_{body} = \Omega\left[-\sin\Omega t \; \hat{e}_x + \cos\Omega t \; \hat{e}_y\right]$

$$= \Omega\left[-\sin\Omega t\left(\hat{e}^\circ_x \cos\omega t + \hat{e}^\circ_y \sin\omega t\right) + \cos\Omega t\left(-\hat{e}^\circ_x \sin\omega t + \hat{e}^\circ_y \cos\omega t\right)\right]$$

$$= \Omega\left[-\sin(\Omega+\omega)t \; \hat{e}^\circ_x + \cos(\Omega+\omega)t \; \hat{e}^\circ_y\right]$$

$$\neq \left(\frac{d\vec{A}}{dt}\right)_{inert}$$

Differ by $\sum_i A_i \frac{d\hat{e}_i}{dt}$:

$$\frac{d\hat{e}_x}{dt} = \omega\left[-\sin\omega t \; \hat{e}^\circ_x + \cos\omega t \; \hat{e}^\circ_y\right]$$

$$\frac{d\hat{e}_y}{dt} = \omega\left[-\cos\omega t \; \hat{e}^\circ_x - \sin\omega t \; \hat{e}^\circ_y\right]$$

So $\sum_i A_i \frac{d\hat{e}_i}{dt} = \cos\Omega t\,[-\omega\,\hat{e}_x^\circ \sin\omega t + \omega\,\hat{e}_y^\circ \cos\omega t]$
$+ \sin\Omega t\,[-\omega\,\hat{e}_x^\circ \cos\omega t - \omega\,\hat{e}_y^\circ \sin\omega t)$

$= \omega\,[-\hat{e}_x^\circ \cos(\Omega+\omega)t + \hat{e}_y^\circ \sin(\Omega+\omega)t]$

So $\left(\frac{d\vec{A}}{dt}\right)_{body} + \sum_i A_i \frac{d\hat{e}_i}{dt} = (\Omega+\omega)[-e_x^\circ \cos(\Omega+\omega)t$
$+ \hat{e}_y^\circ \sin(\Omega+\omega)t]$

$= \left(\frac{d\vec{A}}{dt}\right)_{inert}$ ✓

Important points: $\vec{A}$, $\left(\frac{d\vec{A}}{dt}\right)_{inert}$, $\left(\frac{d\vec{A}}{dt}\right)_{body}$
are all well-defined vectors.
Can express any of them in any frame

Work on simplifying $\frac{d\hat{e}_i}{dt}$

$$\frac{d\hat{e}_i}{dt} = \sum_j \left(\frac{d\hat{e}_i}{dt}\cdot\hat{e}_j\right)\hat{e}_j$$

Define $\frac{d\hat{e}_i}{dt}\cdot\hat{e}_j = \omega_{ij} = \omega_{ij}(t)$

So $\frac{d\hat{e}_i}{dt} = \sum_j \omega_{ij}\,\hat{e}_j$

However, we require $\hat{e}_i\cdot\hat{e}_j = \delta_{ij}$ always

So $\frac{d}{dt}(\hat{e}_i\cdot\hat{e}_j) = 0$
Constrains ω_{ij}:

$$\frac{d}{dt}(\hat{e}_i\cdot\hat{e}_i) = 2\,\hat{e}_i\cdot\frac{d\hat{e}_i}{dt} = \omega_{ii} = 0$$

If $i \neq j$: $\frac{d}{dt}(e_i \cdot e_j) = \frac{d\hat{e}_i}{dt}\hat{e}_j + \hat{e}_i \frac{d\hat{e}_j}{dt}$

$$= \omega_{ij} + \omega_{ji} = 0$$

$$\omega_{ij} = -\omega_{ji}$$

ω_{ij} is antisymmetric matrix: $\begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix}$

Three indep values,
same as a vector

Define $\vec{\omega} = [\omega_1, \omega_2, \omega_3]$

by
$$\omega_1 = \omega_{23}$$
$$\omega_2 = -\omega_{13}$$
$$\omega_3 = \omega_{12}$$

Rewrite $\frac{d\hat{e}_i}{dt} = \sum_j \omega_{ij}\hat{e}_j$ in terms of $\vec{\omega}$:

$$\frac{d\hat{e}_1}{dt} = \omega_{11}\hat{e}_1 + \omega_{12}\hat{e}_2 + \omega_{13}\hat{e}_3$$

$$= 0 + \omega_3\hat{e}_2 - \omega_2\hat{e}_3$$

Similarly, $\frac{d\hat{e}_2}{dt} = -\omega_3\hat{e}_1 + \omega_1\hat{e}_3$

$$\frac{d\hat{e}_3}{dt} = \omega_2\hat{e}_1 - \omega_1\hat{e}_2$$

Express all three as $\frac{d\hat{e}_i}{dt} = \vec{\omega} \times \hat{e}_i$

Good things to know:

If M is antisymmetric 3×3 matrix
and $\vec{c} = M\vec{b}$
can write $\vec{c} = \vec{m}\times\vec{b}$ for $\vec{m}$ defined as above

Can similarly write cross product as matrix

So here, have

$$\boxed{\left(\frac{d\vec{A}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{A}}{dt}\right)_{\text{body}} + \vec{\omega}\times\vec{A}}$$

Identify $\vec{\omega}$ as instantaneous angular velocity of rotating frame

- direction = rotation axis
- magnitude = rotation rate

Note that $\vec{\omega} = \vec{\omega}(t)$ in general

Apply to mechanics: $\vec{A}\to\vec{r}$ = position of particle

$$\text{So}\quad \left(\frac{d\vec{r}}{dt}\right)_{\text{inert}} = \left(\frac{d\vec{r}}{dt}\right)_{\text{body}} + \vec{\omega}\times\vec{r}$$

Relates velocities

Accelerations:

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{inert}} = \left(\frac{d}{dt}\right)_{\text{body}}\left[\left(\frac{d\vec{r}}{dt}\right)_{\text{body}} + \vec{\omega}\times\vec{r}\right] + \vec{\omega}\times\left[\left(\frac{d\vec{r}}{dt}\right)_{\text{body}} + \vec{\omega}\times\vec{r}\right]$$

$$= \left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{body}} + 2\,\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{\text{body}} + \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$\left(\frac{d\vec{\omega}}{dt}\right)_{\text{body}} = \left(\frac{d\vec{\omega}}{dt}\right)_{\text{inert}}$

We know $m\left(\frac{d^2\vec r}{dt^2}\right)_{inert} = \vec F$ total force

So
$$\boxed{m\left(\frac{d^2\vec r}{dt^2}\right)_{body} = \vec F - 2m\vec\omega\times\left(\frac{d\vec r}{dt}\right)_{body} - m\vec\omega\times(\vec\omega\times\vec r) - m\frac{d\vec\omega}{dt}\times\vec r}$$

Newton's Law in rotating frame

Extra terms look like forces

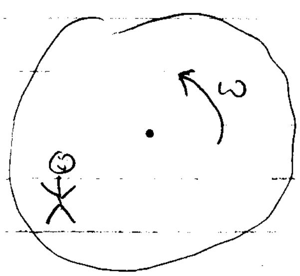
Special names

$-2m\vec\omega\times\left(\frac{d\vec r}{dt}\right)$ = Coriolis force

$-\vec\omega\times(\vec\omega\times\vec r)$ = centrifugal force

Consider each term

Simple picture: child on merry-go-round

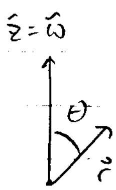


Centrifugal force!

Pushes child away from center

$$\vec F_{cent} = -\vec\omega\times(\vec\omega\times\vec r)$$

Express in cylindrical coords, for $\hat z \propto \vec\omega$



$$\vec\omega\times\vec r = \omega r\sin\theta\ \hat\otimes$$
$$\vec\omega\times(\vec\omega\times\vec r) = \omega^2 r\sin\theta\ \underbrace{\hat z\times\hat\otimes}_{\leftarrow\ \text{vector}}$$

Have $\rho = r\sin\theta$ $\quad\hat\rho = \rightarrow$

So $\vec\omega\times(\vec\omega\times\vec r) = -\omega^2\rho\hat\rho$

$\boxed{\vec F_{cent} = m\omega^2\rho\hat\rho}$ Push away from axis of rotation

Coriolis:

If child throws ball, path seems to curve

$$\vec{F}_{cor} = -2m\,\vec{\omega}\times\vec{v} \qquad (\vec{v} = (\vec{v})_{body})$$

Appears when move $\perp$ to $\vec{\omega}$

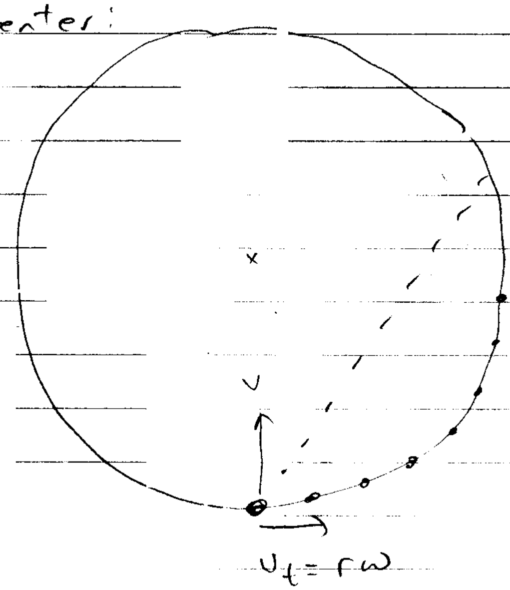
Direction $\perp$ to $\vec{v}$ and $\vec{\omega}$

Example, throw ball towards center:

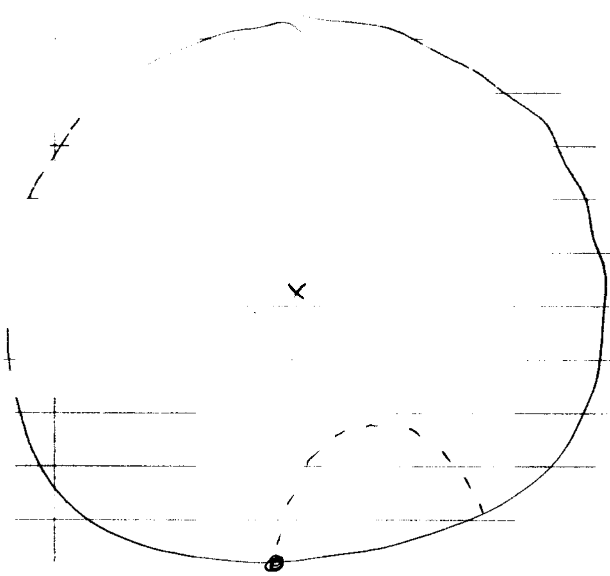
Inertial frame:

In inertial frame, moves in straight line

Not toward center due to intial v_t



Rotating frame



Check direction!

$\vec{F}_{cor} \propto -\vec{\omega}\times\vec{v}$

$\vec{\omega} = \odot$

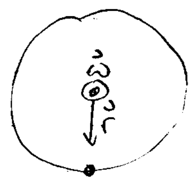
$\vec{v} = \uparrow$

$\vec{\omega}\times\vec{v} = \leftarrow$

$-\vec{\omega}\times\vec{v} = \rightarrow$, correct

Finally, $\quad -m \dfrac{d\vec{\omega}}{dt} \times \vec{r}$

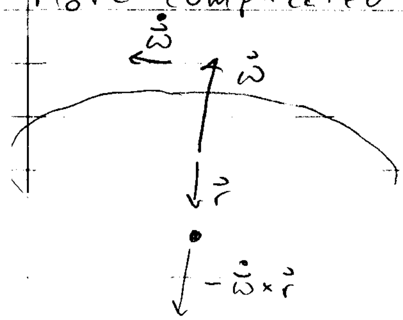
Effect: if merry-go-round speeding up, $\quad \dfrac{d\vec{\omega}}{dt} \propto \odot$



$$\frac{d\vec{\omega}}{dt} \times \vec{r} = \odot \times \downarrow = \rightarrow$$
$$-\frac{d\vec{\omega}}{dt} \times \vec{r} = \leftarrow$$

This is just reaction force for <u>tangential acceleration</u>

More complicated if direction of $\vec{\omega}$ changing



If $\vec{\omega}$ moving $\leftarrow$
Force points $\downarrow$

Because mass needs to accelerate up to stay on platform

In general, get one more extra force
What if body frame also accelerating?

Say origin of body frame at $\vec{R}$ relative to inertial frame

So $\quad \vec{r}_0 = \vec{R} + \vec{r}$

$\vec{r}_0$: position in inertial frame

$\vec{r}$: position in body frame

$$\left(\frac{d^2\vec{r}_0}{dt^2}\right)_{inert} = \left(\frac{d^2\vec{R}}{dt^2}\right)_{inert} + \left(\frac{d^2\vec{r}}{dt^2}\right)_{inert}$$

Or $\left(\frac{d^2\vec{r}_0}{dt^2}\right)_{inert} = \left(\frac{d^2\vec{R}}{dt^2}\right)_{inert} + \left(\frac{d^2\vec{r}}{dt^2}\right)_{body}$

$$+ 2\vec{\omega}\times\frac{d\vec{r}}{dt} + \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$= \frac{1}{m}\vec{F}$$

So complete equation of motion is

$$\boxed{m\left(\frac{d^2\vec{r}}{dt^2}\right)_{body} = \vec{F} - m\left(\frac{d^2\vec{R}}{dt^2}\right)_{inert} - 2m\,\vec{\omega}\times\vec{v} - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r}) - m\frac{d\vec{\omega}}{dt}\times\vec{r}}$$

$\frac{d^2\vec{R}}{dt^2}$ term is usual reaction force

pushes you back in seat when car accelerates

Note: Book uses $\vec{a}$ instead of $\vec{R}$ for position of origin

No good! Easy to think $\vec{a}$ = acceleration

So I'll use $\vec{R}$: don't get confused!

$\Rightarrow$

Again, delay midterm one week?