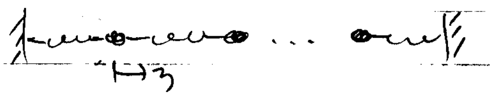
Lecture 10

First, demo from Don?

Recall linear chain



N masses

Found equation of motion

$$m\ddot{y}_j + 2ky_j - k(y_{j+1} - y_{j-1}) = 0$$

Solution $\quad y_j = \sum_{n=1}^{N} C^{(n)} \rho_j^{(n)} \cos(\omega_n t + \phi_n)$

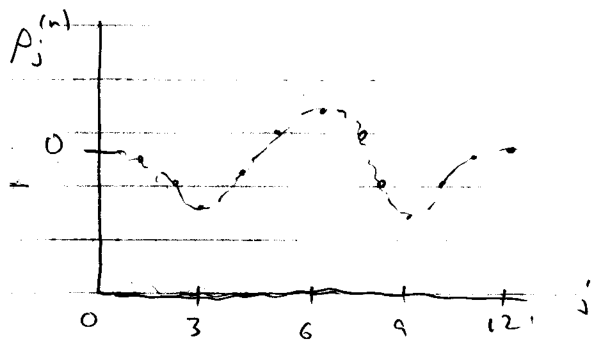
$$\rho_j^{(n)} = A \sin\left(\frac{\pi n j}{N+1}\right)$$

$$\omega_n^2 = \frac{2k}{m}\left(1 - \cos\frac{n\pi}{N+1}\right)$$

$$= \frac{4k}{m} \sin^2 \frac{\pi}{2} \frac{n}{N+1}$$

Solutions look like waves:

$N = 11, \quad n = 3$



In fact, this analysis applies to any type of linear chain of equal masses w/ equal couplings

For instance, transverse motion of masses on a string

→ Look like vibrational modes

Today: go to continuous limit

Could just take $N \to \infty$, $m \to 0$ limit of discrete result

But nicer to get continuous limit directly

Consider general problem of a string:

Characterize by deviation $u(x)$

tension $\tau(x)$ - establishes equilibrium $u(x)=0$

mass density $\sigma(x)$

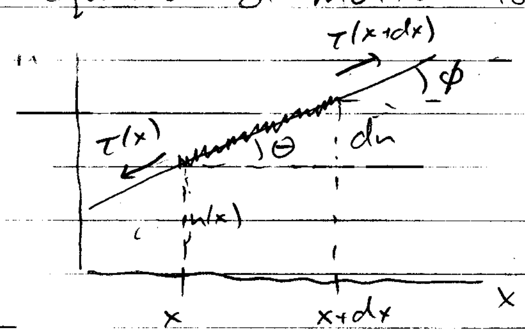
mass/unit length

Note, x is no longer a dynamical variable

$x \neq x(t)$

x is independent, like t (or j)

Get equation of motion for mass element dm between x & $x+dx$



$dm = \sigma\, dx$

Newton says $\quad \sigma\, dx \dfrac{\partial^2 u}{\partial t^2} = \tau(x+dx)\sin\phi - \tau(x)\sin\theta$

For small deviations, θ & ϕ are small:

$$\sin\theta \approx \theta \approx \frac{(du)}{(dx)} = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} u(x,t)$$

and $\sin\phi \approx \phi \approx \frac{\partial}{\partial x} u(x+dx, t)$

So

$$\sigma\, dx\, \frac{\partial^2 u}{\partial t^2} \approx \tau(x+dx) \frac{\partial}{\partial x} u(x+dx) - \tau(x) \frac{\partial}{\partial x} u(x)$$

$$= f(x+dx) - f(x)$$

$$f(x) = \tau \frac{\partial u}{\partial x}$$

$$= \frac{\partial f}{\partial x}\, dx$$

Conclude that

$$\boxed{\sigma \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[\tau \cdot \frac{\partial u}{\partial x}\right]}$$

"String equation" — general equation of motion for continuous string

Simplifies if σ, τ = constant:

$$\frac{\sigma}{\tau} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

= Wave equation $\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$

for $c = \sqrt{\frac{\tau}{\sigma}}$

Relate back to discrete model:

$$u(x) \to z_j \qquad x = ja$$

$$\frac{\partial u}{\partial x} \to \frac{z_{j+1} - z_j}{a}$$

$$\frac{\partial^2 u}{\partial x^2} \to \frac{1}{a}\left[\frac{z_{j+1} - z_j}{a} - \frac{z_j - z_{j-1}}{a}\right]$$

$$= \frac{1}{a^2}\left(z_{j+1} + z_{j-1} - 2z_j\right)$$

and $\sigma a \to m$

So $$\frac{m}{a\tau}\frac{\partial^2 z_j}{\partial t^2} = \frac{1}{a^2}\left(z_{j+1} + z_{j-1} - 2z_j\right)$$

$$m\ddot{z}_j + 2\frac{\tau}{a}z_j - \frac{\tau}{a}\left(z_{j+1} + z_{j-1}\right) = 0$$

like linear chain, $k = \frac{\tau}{a}$

Look for solutions to continuous equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Try separation of variables $u(x,t) = \rho(x)\, g(t)$

$$\rho \ddot{g} = c^2 \rho'' g$$

$$\underbrace{\frac{\ddot{g}}{g}}_{\text{depends on } t} = c^2 \underbrace{\frac{\rho''}{\rho}}_{\text{depends on } x}$$

Both sides must be constant, $= -\omega^2$

So $\ddot{g} = -\omega^2 g \Rightarrow g(t) = \cos(\omega t + \phi)$

$$\rho'' = -\frac{\omega^2}{c^2}\rho$$

$$\rho(x) = \cos(kx+\alpha) \qquad \text{for } k = \frac{\omega}{c}$$

(not spring constant k!)

Allowed values of k and ω are set by boundary conditions

For instance, say $u(0,t) = u(\ell,t) = 0$ (fixed ends)

Then $\rho(0) = \rho(\ell) = 0$

$x=0$: $\cos\alpha = 0 \Rightarrow \alpha = -\frac{\pi}{2}$

$$\rho(x) = \sin kx$$

$x=\ell$ $\quad \sin k\ell = 0 \quad \boxed{k = \frac{n\pi}{\ell}} \quad n = 1, 2, \ldots$

Compare discrete solution $\rho_j = \sin\theta j$ $\quad \theta = \frac{n\pi}{N+1}$

if $x = ja$, $\quad k = \frac{\theta}{a}$

$$= \frac{n\pi}{(N+1)a}$$

$$= \frac{n\pi}{\ell}$$

for total length $\ell = (N+1)a$

Solutions have same form, but $N\to\infty$ ... makes sense

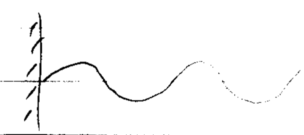
Can define wavelength $\lambda = \frac{2\pi}{k}$

$$\rho = \sin \frac{2\pi x}{\lambda}, \quad \lambda = \text{period in } x$$

See $\lambda = 2\pi \left(\frac{\ell}{n\pi} \right) = \frac{\ell}{2n}$

So $\boxed{n \frac{\lambda}{2} = \ell}$

→ fit half-integer # of wavelengths on string



Again, makes sense.

Finally, write out general solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n \rho_n(x) \cos(\omega_n t + \phi_n)$$

More intersting things to say... come back to this topic in Ch 7

For now, fit it into Lagrangian formalism

Got eqn of motion from Newton
Show how to use Lagrangian instead

Construct L for string:

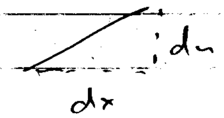
Element dm has velocity $\frac{\partial u}{\partial t}$

Kinetic energy $\frac{1}{2} dm \left(\frac{\partial u}{\partial t} \right)^2$

where $dm = \sigma dx$

Sum over all x: $\boxed{T = \frac{1}{2}\int \sigma \left(\frac{\partial u}{\partial t}\right)^2 dx}$

Potential energy is work done in stretching string



length of element dm is $ds = \sqrt{dx^2 + du^2}$

$$\text{Stretched by } ds - dx$$

$$= dx\left[\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} - 1\right]$$

For small deviations, $\frac{\partial u}{\partial x} \ll 1$, so $\sqrt{} \to 1 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2$

$$ds - dx \simeq \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

Work done in stretching string is $dW = \tau(ds - dx)$

$$V = \int dW = \int \frac{1}{2}\tau\left(\frac{\partial u}{\partial x}\right)^2 dx$$

Combine with T, get

$$\boxed{L = T - V = \frac{1}{2}\int_0^\ell \sigma\left(\frac{\partial u}{\partial t}\right)^2 - \tau\left(\frac{\partial u}{\partial x}\right)^2 dx}$$

Useful to define "Lagrangian density" $\mathcal{L}(x,t)$

$$\text{where } L = \int \mathcal{L}\, dx$$

$$\text{here } \mathcal{L} = \frac{1}{2}\left[\sigma\left(\frac{\partial u}{\partial t}\right)^2 - \tau\left(\frac{\partial u}{\partial x}\right)^2\right]$$

Let's write $u_t = \frac{\partial u}{\partial t}$ $\quad u_x = \frac{\partial u}{\partial x}$

$$\mathcal{L} = \frac{1}{2}\sigma u_t^2 - \frac{1}{2}\tau u_x^2$$

Get eqn of motion by varying coordinates

Know $\delta S = \delta \int_{t_1}^{t_2} L\, dt = 0$

Here coordinates $\rightarrow$ function $u(x,t)$

So, let $u(x,t) \rightarrow u + \delta u(x,t)$

Require $\delta u(x,t_1) = \delta u(x,t_2) = 0$

also require $\delta u(0,t) = \delta u(l,t) = 0$
to maintain boundary condition

In general: $\mathcal{L} = \mathcal{L}(u, u_t, u_x, x, t)$

$$\delta S = \int_{t_1}^{t_2} dt \int_0^l dx \left[\frac{\partial \mathcal{L}}{\partial u}\delta u + \frac{\partial \mathcal{L}}{\partial u_t}\delta u_t + \frac{\partial \mathcal{L}}{\partial u_x}\delta u_x\right]$$

(x + t are independent, don't vary)

Think about

$$\delta u_x = \delta \frac{\partial u}{\partial x}$$

$$= \left[\frac{\partial}{\partial x}(u+\delta u)\right] - \left[\frac{\partial u}{\partial x}\right]$$

$$= \frac{\partial}{\partial x}\delta u$$

So term $\int_{t_1}^{t_2} dt \int_0^l dx \frac{\partial \mathcal{L}}{\partial u_x}\delta u_x = \int dt \int dx \frac{\partial \mathcal{L}}{\partial u_x}\frac{\partial}{\partial x}\delta u$

Integrate by parts in x

$$= \int dt \Big[\underbrace{\frac{\partial \mathcal{L}}{\partial u_x} \delta u \Big|_0^\ell}_{=0 \text{ on } \cancel{\text{boundary}}} - \int dx \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial u_x} \Big]$$

Same for u_t term:

$$\int dt \int dx \frac{\partial \mathcal{L}}{\partial u_t} \delta u_t = - \int dt \int dx \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial u_x} \right) \delta u$$

All together, set

$$\int_{t_1}^{t_2} dt \int_0^\ell dx \left[\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial u_x} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial u_t} \right] \delta u = 0$$

Since δu is arbitrary, this requires

$$\boxed{\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial u_x} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial u_t} = 0}$$

Equation of motion for arbitrary 1D continuous system

So apply to string:

$$\mathcal{L} = \frac{1}{2} \sigma u_t^2 - \frac{1}{2} \tau u_x^2$$

$$\frac{\partial \mathcal{L}}{\partial u} = 0 \qquad \frac{\partial \mathcal{L}}{\partial u_x} = -\tau u_x \qquad \frac{\partial \mathcal{L}}{\partial u_t} = \sigma u_t$$

so $\quad - \frac{\partial}{\partial x} (-\tau u_x) - \frac{\partial}{\partial t} (\sigma u_t) = 0$

σ independent of t, so

$$\sigma \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[\tau \frac{\partial u}{\partial x}\right]$$ = string equation as before

So Lagrangians work for continuous systems

Convenient if system is constrained
- not quite so common here

But formalism is very important
= basis for quantum field theory

We'll see how regular QM is quantized version of classical mechanics

Quantum field theory = quantized version of classical continuum mechanics

Wraps up Ch 4
but we will return to strings in Ch 7

Next jump to Ch 2, rotating frames