Lecture 9

Last time, went over general procedure to find normal modes

1) Find equilibrium $\frac{\partial V}{\partial q_\sigma} = 0$. Get $m_{\sigma\lambda} = \frac{\partial^2 T}{\partial \dot q_\sigma \partial \dot q_\lambda}$, $V_{\sigma\lambda} = \frac{\partial^2 V}{\partial q_\sigma \partial q_\lambda}$

2) Find mode frequencies $|\omega^2 \mathbb{M} - \mathbb{V}| = 0$

3) Find eigenvectors $\mathbb{V}\vec{\rho}_s = \omega_s^2 \mathbb{M}\vec{\rho}$

Then motion given by

$$q_\sigma(t) = q_\sigma^{(0)} + \sum_s C^{(s)} \rho_\sigma^{(s)} \cos(\omega_s t + \phi_s)$$

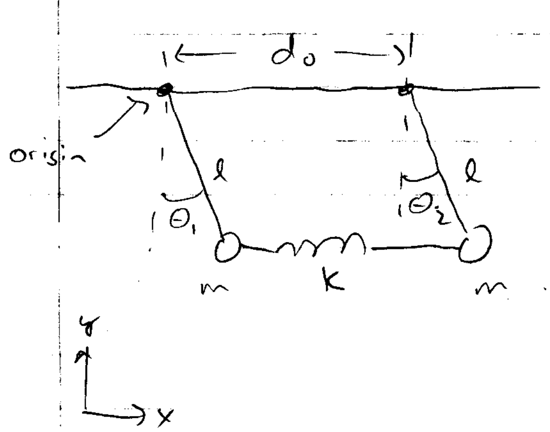
$C^{(s)}$, $\phi^{(s)}$ from initial conditions

This assumes $\omega_s^2 > 0 \Rightarrow$ motion is stable

If $\omega_s^2 < 0$, mode is unstable $q_\sigma \sim e^{\pm |\omega_s| t}$

If $\omega_s^2 = 0$, mode is neutral $q_\sigma \sim A + Bt$

Example: Coupled pendulums



Connected by spring
spring const k
relaxed length d_0

$$T = \frac{1}{2} m l^2 \dot{\theta}_1^2 + \frac{1}{2} m l^2 \dot{\theta}_2^2$$

$$V_{gravity} = -mgl\cos\theta_1 - mgl\cos\theta_2$$

V_{spring}?

Recall $V = \frac{1}{2} k (d - d_0)^2$

d = distance between masses

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$x_1 = l\sin\theta_1 \qquad x_2 = d_0 + l\sin\theta_2$$
$$y_1 = -l\cos\theta_1 \qquad y_2 = -l\cos\theta_2$$

A bit messy. But we know equilibrium is
$\theta_1 = \theta_2 = 0$
So take θ's small and Taylor expand
(to 2^{nd} order)

$$d^2 = (l\theta_1 - l\theta_2 - d_0)^2 + \underbrace{l^2\left[\left(1 - \frac{\theta_1^2}{2}\right) - \left(1 - \frac{\theta_2^2}{2}\right)\right]^2}_{\frac{l^2}{4}\left[\theta_2^2 - \theta_1^2\right]^2 \text{ 4th order!}}$$

So $d^2 \approx (l\theta_1 - l\theta_2 - d_0)^2$

$$d \approx d_0 - l(\theta_1 - \theta_2)$$

and $V_{spring} = \frac{1}{2} k l^2 (\theta_1 - \theta_2)^2$

Similarly, $V_{grav} = -mgl\left(2 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2}\right)$

$$V = V_0 + \frac{1}{2} m g l \left[\theta_1^2 + \theta_2^2 + \frac{kl}{mg} (\theta_1 - \theta_2)^2 \right]$$

Define $\varkappa = \frac{kl}{mg}$

$$\frac{1}{2} m g l \left[(1+\varkappa)\theta_1^2 + (1+\varkappa)\theta_2^2 - 2\varkappa \theta_1 \theta_2 \right]$$

Get $\mathbb{V}$:

$$V_{11} = \frac{\partial^2 V}{\partial \theta_1^2} = m g l (1+\varkappa)$$

$$V_{22} = m g l (1+\varkappa)$$

$$V_{12} = V_{21} = \frac{\partial^2 V}{\partial \theta_1 \partial \theta_2} = -m g l \varkappa$$

Also $\mathbb{M} = \begin{bmatrix} m l^2 & 0 \\ 0 & m l^2 \end{bmatrix}$ $\quad \mathbb{V} = m g l \begin{bmatrix} 1+\varkappa & -\varkappa \\ -\varkappa & 1+\varkappa \end{bmatrix}$

$$\left| \mathbb{M}\omega^2 - \mathbb{V} \right| = \begin{vmatrix} m l^2 \omega^2 - m g l (1+\varkappa) & m g l \varkappa \\ m g l \varkappa & m l^2 \omega^2 - m g l (1+\varkappa) \end{vmatrix}$$

$$= \left[m l^2 \omega^2 - m g l (1+\varkappa) \right]^2 - m^2 g^2 l^2 \varkappa^2 = 0$$

Easy solution:

$$m l^2 \omega^2 - m g l (1+\varkappa) = \pm m g l \varkappa$$

$$\omega^2 = \frac{g}{l}(1+\varkappa) \pm \frac{g}{l}\varkappa$$

$$\boxed{\omega_1^2 = \frac{g}{l}} , \quad \boxed{\omega_2^2 = \frac{g}{l}(1+2\varkappa) = \frac{g}{l} + \frac{2k}{m}}$$

Two distinct frequencies

Eigenvectors:

$s=1$: $\omega_1^2 = \frac{g}{l}$

$$\omega_1^2 M - V = \begin{bmatrix} mgl & 0 \\ 0 & mgl \end{bmatrix} - mgl \begin{bmatrix} 1+\varkappa & -\varkappa \\ -\varkappa & 1+\varkappa \end{bmatrix}$$

$$= mgl \begin{bmatrix} -\varkappa & \varkappa \\ \varkappa & -\varkappa \end{bmatrix}$$

Want $$mgl\varkappa \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = 0$$

$$\Rightarrow \rho_1 = \rho_2$$

$$\tilde{\rho}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

don't worry about normalization unless we need to

$s=2$: $\omega_2^2 = \frac{g}{l}(1+2\varkappa)$:

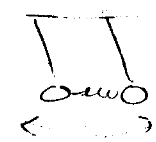
$$\omega_2^2 M - V = mgl \begin{bmatrix} 1+2\varkappa & 0 \\ 0 & 1+2\varkappa \end{bmatrix} - mgl \begin{bmatrix} 1+\varkappa & -\varkappa \\ -\varkappa & 1+\varkappa \end{bmatrix}$$

$$= mgl \begin{bmatrix} \varkappa & \varkappa \\ \varkappa & \varkappa \end{bmatrix}$$

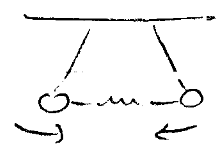
Solve $$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = 0$$

$$\tilde{\rho}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For mode 1, masses move together
Spring never stretches,
so no effect



For mode 2, masses move opposite



Generel solution

$$\theta_1(t) = C^{(1)}\cos(\omega_1 t + \phi_1) + C^{(2)}\cos(\omega_2 t + \phi_2)$$

$$\theta_2(t) = C^{(2)}\cos(\omega_1 t + \phi_1) - C^{(1)}\cos(\omega_2 t + \phi_2)$$

For instance, if $\theta_1(0) = \alpha \qquad \dot{\theta}_1 = \dot{\theta}_2 = \theta_2 = 0$

Then
$$\alpha = C^{(1)}\cos\phi_1 + C^{(2)}\cos\phi_2 \qquad (1)$$
$$0 = \omega_1 C^{(1)}\sin\phi_1 + \omega_2 C^{(2)}\sin\phi_2 \qquad (1)$$
$$0 = C^{(1)}\cos\phi_1 - C^{(2)}\cos\phi_2 \qquad (3)$$
$$0 = \omega_1 C^{(1)}\sin\phi_1 - \omega_2 C^{(2)}\sin\phi_2 \qquad (4)$$

Add $(2)+(4)$: $\quad 2\omega_1 C^{(1)}\sin\phi_1 = 0$
$$\Rightarrow \quad \phi_1 = 0$$

Sustract $(2)-(4)$: $\quad 2\omega_2 C^{(2)}\sin\phi_2 = 0$
$$\Rightarrow \quad \phi_2 = 0$$

So
$$\alpha = C^{(1)} + C^{(2)}$$
$$0 = C^{(1)} - C^{(2)} \qquad \Rightarrow \quad C^{(1)} = C^{(2)} = \frac{\alpha}{2}$$

$$\theta_1(t) = \frac{\alpha}{2}\left(\cos\omega_1 t + \cos\omega_2 t\right)$$

$$\theta_2(t) = \frac{\alpha}{2}\left(\cos\omega_1 t - \cos\omega_2 t\right)$$

Think about in limit of weak spring.

$\frac{k}{m} << \frac{g}{\ell}$ or $\varkappa << 1$

Then $\omega_2 = \omega_1 \sqrt{1+2\varkappa} \approx \omega_1 (1+\varkappa)$

$$\cos\omega_1 t + \cos\omega_2 t \rightarrow 2\cos\frac{(\omega_2+\omega_1)t}{2}\cos\frac{(\omega_2-\omega_1)t}{2}$$

$$= 2\cos\omega_1 t \cos\frac{\Delta t}{2}$$

$$\omega_1 = \sqrt{\frac{g}{\ell}}, \quad \Delta = \varkappa\omega_0$$

$$\cos\omega_1 t - \cos\omega_2 t \rightarrow 2\sin\frac{(\omega_2+\omega_1)t}{2}\sin\frac{(\omega_2-\omega_1)t}{2}$$

$$= 2\sin\omega_1 t \sin\frac{\Delta t}{2}$$

So $$\begin{bmatrix}\Theta_1 \\ \Theta_2\end{bmatrix} = \alpha\begin{bmatrix}\cos\omega_1 t \cos\frac{\Delta t}{2} \\ \sin\omega_1 t \sin\frac{\Delta t}{2}\end{bmatrix}$$

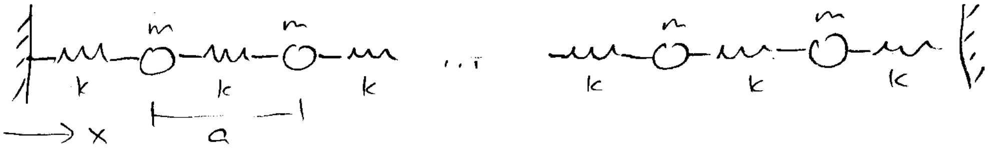
Each mass oscillates at ω_1 but amplitude modulated at $\Delta/2$

Energy trades back and forth between modes
$\Rightarrow$ Generic result for two-mode system

In contrast, if start with excitation of single normal mode, get steady oscillation

See in demo!

Let's look at a bigger problem:
Linear chain of N masses and springs



Take equal masses m, constants k
fixed end points, equil length a
motion along x only

Define η_j = displacement of mass j from equilibrium

$$x_j = x_j^{(0)} + \eta_j \qquad x_j^{(0)} = ja$$

Get $$T = \frac{1}{2} m \sum_{j=1}^{N} \dot{\eta}_j^2$$

Potential $$V_{j,j+1} = \frac{1}{2} k \left(x_{j+1} - x_j - a\right)^2$$
$$= \frac{1}{2} k \left(\eta_{j+1} - \eta_j\right)^2$$

What about ends?
First spring gives $\frac{1}{2} k \eta_1^2$
Last $\frac{1}{2} k \eta_N^2$

Define $\eta_0 = 0$ and $\eta_{N+1} = 0$

Then $$U = \frac{1}{2} k \sum_{j=0}^{N} \left(\eta_{j+1} - \eta_j\right)^2$$

$$L = \frac{1}{2} m \sum_{j=0}^{N} \dot{\eta}_j^2 - \frac{1}{2} k \sum_{j=0}^{N} \left(\eta_{j+1} - \eta_j\right)^2$$

Gives $\mathbb{M}, \mathbb{V}$ = N x N matrices

Can solve for eigenvalues by recursion
book demonstrates

But really easier here to just tackle eqn of motion

$$\frac{\partial L}{\partial \dot{z}_j} = m\dot{z}_j \qquad \frac{\partial L}{\partial z_j} = -k(z_j - z_{j-1}) + k(z_{j+1} - z_j)$$

So $$\boxed{m\ddot{z}_j + 2k z_j - k(z_{j+1} + z_{j-1}) = 0}$$

$j = 1$ to N

Try solution $z_j = \rho_j \cos(\omega t + \phi)$

$$(-m\omega^2 + 2k)\rho_j - k(\rho_{j+1} + \rho_{j-1}) = 0$$

Relates ρ_j for different js
Similar to how ODE relates $f(x)$ at different x's

Here like ODE with constant coeffs, try
similar solution

$$\rho_j = A e^{i\theta j}$$

Then $$(2k - m\omega^2) A e^{i\theta j} - kA\left(e^{i\theta(j+1)} + e^{i\theta(j-1)}\right) = 0$$

$$2k - m\omega^2 - k\left(e^{i\theta} + e^{-i\theta}\right) = 0$$
$$- 2k\cos\theta = 0$$

So $$\boxed{\omega^2 = \frac{2k}{m}(1 - \cos\theta)}$$

Get possible values of Θ from boundary condition

Note Θ and $-\Theta$ give same ω^2

So for given ω, have $\rho_j = A e^{i\Theta j} + B e^{-i\Theta j}$

But require $\rho_0 = \rho_{N+1} = 0$

$$A + B = 0 \quad \Rightarrow \quad B = -A$$

$$A e^{i(N+1)\Theta} - A e^{-i(N+1)\Theta} = 0$$

$$2iA \sin(N+1)\Theta = 0$$

Requires $\Theta = \frac{n\pi}{N+1}$ $\quad n = 1$ to N

($n=0$ gives $\rho_j = 0$ for all j, not a mode
$n > N$ repeats same Θ's as have)

So mode frequencies are

$$\omega_n^2 = \frac{2k}{m}\left(1 - \cos\frac{n\pi}{N+1}\right) \qquad n = 1 \text{ to } N$$

$$\boxed{\omega_n^2 = \frac{4k}{m} \sin^2 \frac{\pi}{2}\frac{n}{N+1}}$$

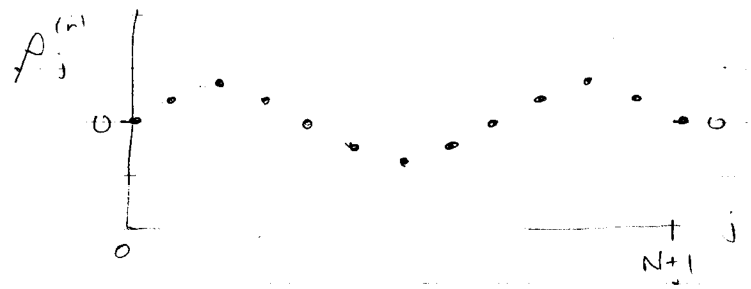
Eigenvectors

$$\rho_j^{(n)} = A\left(e^{i\Theta j} - e^{-i\Theta j}\right).$$

$$\boxed{\rho_j^{(n)} = A' \sin\left(\frac{\pi n j}{N+1}\right)}$$

A' = normalization factor

Look like waves:



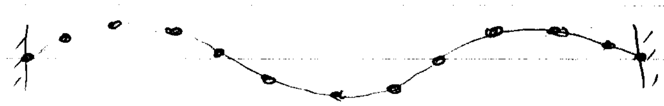
N = 11   n = 3

Full solution

$$\eta_j(t) = \sum_{n=1}^{N} C^{(n)} \sin\frac{\pi n j}{N+1} \cos(\omega_n t + \phi_n)$$

This solution applies to any linear chain of N equal masses with equal couplings

For instance, transverse motion of N masses on a string



Start to look like vibrational modes of a string ... pursue further next time.