Lecture 8

Last time wrapped up Ch 3
Should be able to use Lagrangians for simple systems

Now generalize to complex systems!
many particles

Consider system with n coords

Assume no time varying constraints

Then know from last time $T = \frac{1}{2} \sum_{\nu\lambda} m_{\nu\lambda} \dot{q}_\nu \dot{q}_\lambda$

where $m_{\nu\lambda}(q's) = \sum_i m_i \frac{\partial x_i}{\partial q_\nu} \frac{\partial x_i}{\partial q_\lambda}$

Also have $V(q')$

Eqn of motion $\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} + \frac{\partial U}{\partial q_\sigma} = 0$

Look for static equilibrium solutions $\ddot{q}_\sigma = \dot{q}_\sigma = 0$

Then $T=0$, reduces to $\frac{\partial U}{\partial q_\sigma} = 0$

Recall $-\frac{\partial U}{\partial q_\sigma} = Q_\sigma$ generalized force

Makes sense, equilibrium where force $= 0$

So only need U

Find solution $q_\sigma = q_\sigma^0$ to $\frac{\partial U}{\partial q_\sigma} = 0$

Then consider small displacement $q_\sigma = q_\sigma^0 + \eta_\sigma$

Get $T \to \frac{1}{2} \sum_{\sigma\lambda} m_{\sigma\lambda}(q^0)\, \dot{\eta}_\sigma \dot{\eta}_\lambda$ to lowest order

$m_{\sigma\lambda}(q^0)$ = constant, real, symmetric matrix

Also have $V = V(q_1^0 + \eta_1, \ldots q_n^0 + \eta_n)$

Taylor expand!

$$V \simeq V(q^0) + \underbrace{\sum_\sigma \frac{\partial V}{\partial q_\sigma}\Big|_{q^0}}_{=0 \text{ by def } q^0} \eta_\sigma + \frac{1}{2} \sum_{\sigma\lambda} \frac{\partial^2 V}{\partial q_\sigma \partial q_\lambda}\Big|_{q_0} \eta_\sigma \eta_\lambda$$

Define $V_0 = V(q^0)$ $\qquad V_{\sigma\lambda} = \frac{\partial^2 V}{\partial q_\sigma \partial q_\lambda}\Big|_{q^0}$

So $V \simeq V_0 + \frac{1}{2} \sum_{\sigma\lambda} V_{\sigma\lambda} \eta_\sigma \eta_\lambda$

$V_{\sigma\lambda}$ = constant, real, symmetric matrix "potential matrix"

Put together: $L = T - V$

$$= \frac{1}{2} \sum_{\sigma\lambda} \left(m_{\sigma\lambda} \dot{\eta}_\sigma \dot{\eta}_\lambda - V_{\sigma\lambda} \eta_\sigma \eta_\lambda \right)$$

Think of η's as new coords

Eqn of motion $\frac{d}{dt} \frac{\partial L}{\partial \dot{\eta}_\sigma} - \frac{\partial L}{\partial \eta_\sigma} = 0$

$$\frac{\partial L}{\partial \dot{\eta}_\sigma} = \sum_\lambda m_{\sigma\lambda} \dot{\eta}_\lambda \qquad \frac{\partial L}{\partial \eta_\sigma} = -\sum_\lambda V_{\sigma\lambda} \eta_\lambda$$

So $$\boxed{\sum_\lambda m_{\sigma\lambda}\ddot{\eta}_\lambda + V_{\sigma\lambda}\eta_\lambda = 0}$$.

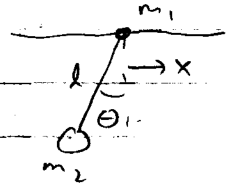
In matrix notation $m_{\sigma\lambda} \to \mathbb{M}$

$V_{\sigma\lambda} \to \mathbb{V}$

$\eta_\sigma \to \vec{\eta}$

$$\boxed{\mathbb{M}\ddot{\vec{\eta}} + \mathbb{V}\vec{\eta} = 0}$$

Example: HW 3.3 - sliding pendulum



$$V = -m_2 g l \cos\theta$$

Equilibrium at $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial \theta} = 0$

$$\sin\theta = 0$$

$$\Rightarrow \theta = 0, \pi; \quad x = \text{arb}$$

Consider $\theta = 0$

From HW, $T = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 + m_2 \dot{x}\dot{\theta} l \cos\theta$

Get mass matrix

In general, have $m_{\sigma\lambda} = \frac{\partial^2 T}{\partial q_\sigma \partial q_\lambda}$

Here $m_{xx} = \frac{\partial^2 T}{\partial \dot{x}^2} = m_1 + m_2$

$$m_{\theta\theta} = \frac{\partial^2 T}{\partial \dot{\theta}^2} = m_2 l^2$$

$$m_{x\theta} = m_{\theta x} = \frac{\partial^2 T}{\partial \dot{x} \partial \dot{\theta}} = m_2 l \cos\theta \Big|_{\theta=0} = m_2 l$$

So $\mathbb{M} = \begin{bmatrix} m_1+m_2 & m_2 l \\ m_2 l & m_2 l^2 \end{bmatrix}$

Get potential matrix $V_{\sigma\lambda} = \frac{\partial^2 V}{\partial q_\sigma \partial q_\lambda}$

$V_{xx} = V_{x\theta} = 0$

$V_{\theta\theta} = \frac{\partial^2 V}{\partial \theta^2} = m_2 g l \cdot \cos\theta = m_2 g l$

$\mathbb{V} = \begin{bmatrix} 0 & 0 \\ 0 & m_2 g l \end{bmatrix}$

How to solve $\mathbb{M}\ddot{\vec{z}} + \mathbb{V}\vec{z} = 0$

Try solution $\vec{z} = \vec{\rho}\cos(\omega t + \phi)$

$\ddot{\vec{z}} = -\omega^2 \vec{\rho}\cos(\omega t + \phi)$

So need $(-\omega^2 \mathbb{M} + \mathbb{V})\vec{\rho} = 0$

General problem $\mathbb{A}\vec{\rho} = 0$ has solution only if $|\mathbb{A}| = 0$

Here require $\boxed{|\mathbb{M}\omega^2 - \mathbb{V}| = 0}$

Pendulum example:

$$\mathbb{M}\omega^2 - \mathbb{V} = \begin{bmatrix} \omega^2(m_1+m_2) & \omega^2 m_2 l \\ \omega^2 m_2 l & \omega^2 m_2 l^2 - m_2 g l \end{bmatrix}$$

$$|\quad| = \omega^2(m_1+m_2)(\omega^2 m_2 l^2 - m_2 g l) - \omega^4 m_2^2 l^2$$

One solution $\boxed{\omega_1^2 = 0}$

Leaves $(m_1+m_2)(\omega^2 l - g) - \omega^2 m_2 l = 0$

$$\omega^2 m_1 l - (m_1+m_2) g = 0$$

$$\boxed{\omega_2^2 = \frac{m_1+m_2}{m_1}\,\frac{g}{l}}$$

In general, get n^{th} order polynomial in ω^2

n roots ω_s^2 $s = 1$ to n

Each with eigenvector $\vec{\rho}_s = \rho_\sigma^{(s)}$

In our example:

$s=1$ $\omega_s^2 = 0$: $\begin{bmatrix} 0 & 0 \\ 0 & m_2 g l \end{bmatrix}\begin{bmatrix} \rho_1^{(1)} \\ \rho_2^{(1)} \end{bmatrix} = 0$

$$\vec{\rho}_1 = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{arb } \alpha$$

$s=2$ $\omega_2^2 = \frac{m_1+m_2}{m_1}\frac{g}{l}$

$$\omega^2 M - V = \begin{bmatrix} \frac{(m_1+m_2)^2}{m_1}\frac{g}{l} & \frac{m_2(m_1+m_2)}{m_1} g \\ \frac{m_2(m_1+m_2)}{m_1} g & \frac{m_2}{m_1} g l \end{bmatrix}\begin{bmatrix} \rho_1^{(2)} \\ \rho_2^{(2)} \end{bmatrix} = 0$$

First row $\frac{(m_1+m_2)^2}{m_1}\frac{g}{l}\rho_1^{(2)} + \frac{m_2(m_1+m_2)}{m_1} g \rho_2^{(2)} = 0$

$$(m_1+m_2)\rho_1^{(2)} + m_2 l \rho_2^{(2)} = 0$$

Solution $\begin{bmatrix}\rho_1^{(1)} \\ \rho_2^{(1)}\end{bmatrix} = \beta\begin{bmatrix}-m_2 l \\ m_1+m_2\end{bmatrix}$ arb β

Would get same result from 2nd row

So everything works pretty much the same as in regular eigenvalue problem

$$\mathbb{H}\vec{v} = \lambda\vec{v}$$

In fact, can reduce two-matrix problem to regular one

First prove lemma:

Can write mass matrix $\mathbb{M} = \mathbb{B}^{+}\mathbb{B}$

* Not following text!

New real matrix $\mathbb{B}$; $\mathbb{B}^{+}$ = transpose

Proof: • $\mathbb{M}$ is real and symmetric: can diagonalize

$$\mathbb{M} = \mathbb{S}^{+}\underset{\underset{\text{diagonal}}{\uparrow}}{\mathbb{D}}\,\mathbb{S}$$

- all eigenvalues of $\mathbb{M}$ are non-negative: If one λ were <0, motion for corresponding eigenvector would have $T<0$

Since $T = \frac{1}{2} m \vec{\rho}^{+}\mathbb{M}\vec{\rho}$

$$= \frac{1}{2} m \vec{\rho}^{+} \lambda \vec{\rho}$$

$$= \frac{1}{2} m \lambda |\rho|^2$$

So can write $\mathbb{D} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} = \mathbb{Q}^{+}\mathbb{Q}$

for $\mathbb{Q} = \mathbb{Q}^{+} = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \sqrt{\lambda_2} & \\ & & \ddots \end{bmatrix}$ real

So $\mathbb{M} = \mathbb{S}^{+}\mathbb{Q}^{+}\mathbb{Q}\mathbb{S} = \mathbb{B}^{+}\mathbb{B}$

for $\mathbb{B} = \mathbb{Q}\mathbb{S}$ $\square$

(note $\mathbb{B}$ is not orthonormal, $\mathbb{B}^{+}\mathbb{B} \neq \mathbb{1}$)

Apply to eigenvalue problem:

$$\mathbb{V}\vec{\rho} = \omega^2 \mathbb{M}\vec{\rho}$$

$$= \omega^2 \mathbb{B}^{+}\mathbb{B}\vec{\rho}$$

$$(\mathbb{B}^{+})^{-1}\mathbb{V}\,\mathbb{B}^{-1}\mathbb{B}\vec{\rho} = \omega^2 \mathbb{B}\vec{\rho}$$

Define $\mathbb{V}' = (\mathbb{B}^{-1})^{+}\mathbb{V}\,\mathbb{B}^{-1}$, $\vec{\rho}\,' = \mathbb{B}\vec{\rho}$

↳ still symmetric!

Then

$\mathbb{V}'\vec{\rho}\,' = \omega^2\vec{\rho}\,'$, regular eigenvalue problem. $\square$

Not really useful for getting ω^2... easiest to just solve $|\mathbb{V} - \omega^2\mathbb{M}| = 0$

But can use results we know from standard problem

General properties (from linear algebra)

1) Eigenvalues of a real symmetric matrix are real

$\Rightarrow \omega^2$ real

2) Eigenvectors of real symmetric matrix can be taken real

$$\Rightarrow \vec{\rho}\,' \text{ real} \Rightarrow \vec{\rho} = \mathbb{B}^{-1}\vec{\rho}\,' \text{ real}$$

3) Eigenvectors corresponding to distinct eigenvalues are orthogonal

$$\vec{\rho}_s^{\,\prime +}\vec{\rho}_t^{\,\prime} = 0 \quad \text{if} \quad \omega_s^2 \neq \omega_t^2$$

$$(\mathbb{B}\vec{\rho}_s)^+(\mathbb{B}\vec{\rho}_t) = 0$$

ω

$$\vec{\rho}_s^{\,+}\mathbb{B}^+\mathbb{B}\vec{\rho}_t = 0$$

$$\boxed{\vec{\rho}_s^{\,+}\mathbb{M}\vec{\rho}_t = 0}$$

In components, $\sum_{\sigma\lambda} \rho_\sigma^{(s)} m_{\sigma\lambda} \rho_\lambda^{(t)} = 0$

Say $\vec{\rho}$'s are orthogonal w/respect to $\mathbb{M}$

Note that if $\omega_s^2 = \omega_t^2$, can still make $\vec{\rho}_s^{\,+}\mathbb{M}\vec{\rho}_t = 0$

use Gramm-Schmit orthogonalization procedure

Also, we can choose normalization so $\vec{\rho}_s^{\,+}\mathbb{M}\vec{\rho}_s = 1$

So in general $\vec{\rho}_s^{\,+} \mathbb{M} \vec{\rho}_t = \delta_{st}$

Note, this gives us matrix $\mathbb{B}$:

Make $\mathbb{A} = \left[\begin{bmatrix} \vec{\rho}_1 \end{bmatrix} \begin{bmatrix} \vec{\rho}_2 \end{bmatrix} \cdots \begin{bmatrix} \vec{\rho}_n \end{bmatrix} \right]$ matrix of eigenvectors

or $a_{\sigma\lambda} = \rho^{(\sigma)}_{\lambda}$

Then $\mathbb{A}^+ \mathbb{M} \mathbb{A} = \mathbb{1}$

$$\mathbb{M} = (\mathbb{A}^+)^{-1} \mathbb{A}^{-1} = \mathbb{B}^+ \mathbb{B}$$

So $\mathbb{B} = \mathbb{A}^{-1}$ $\quad (\neq \mathbb{A}^+)$

$\mathbb{A}$ is pretty useful. Called the "model matrix"

Note that $\mathbb{A}$ diagonalizes $\mathbb{V}$ as well as $\mathbb{M}$:

$$\vec{\rho}_s^{\,+} \mathbb{V} \vec{\rho}_t = \vec{\rho}_s^{\,+} (\omega_t^2 \mathbb{M} \vec{\rho}_t) = \omega_t^2 \delta_{st}$$

So $\mathbb{A}^+ \mathbb{V} \mathbb{A} = \begin{bmatrix} \omega_1^2 & & 0 \\ & \omega_2^2 & \\ 0 & & \ddots \ \omega_n^2 \end{bmatrix}$

Construct $\mathbb{A}$ for our example system

$s=1 \qquad \omega_1^2 = 0 \qquad \vec{\rho}_1 = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Pick α so $\vec{\rho}_1^{\,+} \mathbb{M} \vec{\rho}_1 = 1$

$$\mathbb{M} = \begin{bmatrix} m_1 + m_2 & m_2 l \\ m_2 l & m_2 l^2 \end{bmatrix}$$

Need $\alpha^2 = \frac{1}{m_1+m_2}$, $\alpha = \frac{1}{\sqrt{m_1+m_2}}$

$s=2$! $\omega_2^2 = \left(1+\frac{m_2}{m_1}\right)\frac{g}{\ell}$ $\vec{\rho}_2 = \beta\begin{bmatrix} -m_1\ell \\ m_1+m_2 \end{bmatrix}$

Make $\vec{\rho}_2^{+}\,\mathbb{M}\,\vec{\rho}_2 = 1$

Get $\beta^2 = \frac{1}{(m_1+m_2)m_1m_2\ell^2}$

Finally, $\mathbb{A} = \Big[[\rho_1]\ [\rho_2]\Big]$

$$= \begin{bmatrix} \frac{1}{\sqrt{m_1+m_2}} & -\sqrt{\frac{m_2}{m_1(m_1+m_2)}} \\ 0 & \frac{1}{\ell}\sqrt{\frac{m_1+m_2}{m_1m_2}} \end{bmatrix}$$

diagonalizes $\mathbb{M}$ and $\mathbb{V}$.

Finally, can use $\mathbb{A}$ to define new coordinates

$$\vec{\xi}(t) = \mathbb{A}^{-1}\vec{\zeta}(t) = \mathbb{A}^{+}\mathbb{M}\vec{\zeta}$$

"Normal coordinates" and $\vec{\zeta} = \mathbb{A}\vec{\xi}$

Rewrite L in terms of ξ's:

$$L = \tfrac{1}{2}\left(\dot{\vec{\zeta}}^{+}\mathbb{M}\dot{\vec{\zeta}} - \vec{\zeta}^{+}\mathbb{V}\vec{\zeta}\right)$$

$$= \tfrac{1}{2}\Big(\dot{\vec{\xi}}^{+}\underbrace{\mathbb{A}^{+}\mathbb{M}\mathbb{A}}_{\text{identity}}\dot{\vec{\xi}} - \vec{\xi}^{+}\underbrace{\mathbb{A}^{+}\mathbb{V}\mathbb{A}}_{\text{diagonal}}\vec{\xi}\Big)$$

So $L = \frac{1}{2}\sum_{\sigma}\left(\dot{\xi}_\sigma^2 - \omega_\sigma^2\xi_\sigma^2\right)$ = n uncoupled harmonic oscillators