Lecture 6 - Lagrange Multipliers

Last time, started talking about technique to obtain constraint forces from Lagrange method, when desired.

Approach: include constraint "by hand" rather than implicitly

Suppose Lagrangian $L(\{q_\sigma, \dot{q}_\sigma\}, t)$ $\quad \sigma = 1$ to $n+1$

constraint $\quad f(q_\sigma\text{'s}, t) = 0$

For true path, S is minimum, so

$$\int \sum_\sigma \delta q_\sigma \left[\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \right] dt = 0$$

But δq_σ's are not independent

Can vary δq_σ for $\sigma = 1$ to n

But then δq_{n+1} is set by constraint. $\boxed{\text{$n$ eqns, $n+1$ unknowns}}$

However: also true that $\delta f = \sum_\sigma \frac{\partial f}{\partial q_\sigma} \delta q_\sigma = 0$

for any δq_σ that satisfies constraint

Can multiply by arbitrary fcn $\lambda(q, \dot{q}, t)$, still zero

So we know

$$\int \sum_\sigma \delta q_\sigma \left[\frac{\partial L}{\partial q_\sigma} + \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} + \lambda \frac{\partial f}{\partial q_\sigma} \right] dt = 0$$

Can use independence to say $[\quad] = 0$ for $\sigma = 1$ to n

now pick λ so its true for $\sigma = n+1$ as well

Then $n+1$ eqns, $n+2$ unknowns (q_σ's & λ)

But also have constraint: $n+2$ eqns total

For multiple constraints:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} - \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$$

$$f_j(q) = 0$$

Solution gives constrained motion

How does this give us constraint forces?

Consider physical significance of λ

Suppose we included constraint forces directly in L

Then would have constraint potential $V_c(q)$

$$L \rightarrow T - V - V_c = L^{(0)} - V_c$$

Would have eqn of motion

$$\frac{d}{dt}\frac{\partial L^{(0)}}{\partial \dot{q}_\sigma} - \frac{\partial L^{(0)}}{\partial q_\sigma} + \frac{\partial V_c}{\partial q_\sigma} = 0$$

Compare to above, see that $\lambda \frac{\partial f}{\partial q_\sigma}$ corresponds to $-\frac{\partial V_c}{\partial q_\sigma}$

But $-\frac{\partial V_c}{\partial q_\sigma}$ = "generalized" constraint force Q_σ

or $\boxed{Q_\sigma^{(j)} = \lambda_j \frac{\partial f_j}{\partial q_\sigma}}$

Knowing λ gives constraint force

Relate to Cartesian force F_σ by work:

If motion δq_σ violates constraint, constraint force does work

$$\delta W = Q_\sigma \, \delta q_\sigma = \sum_i F_i \, \delta x_i$$

Given $x_i(q\text{'s})$, can solve for F_i as desired

Example: pendulum, calculate tension

Set up L with r, θ both free

$$L = \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + mgr\cos\theta$$

Constraint $f(r,\theta) = r - l = 0$

Eqns of motion

r: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{r}} = m\ddot{r}$ $\qquad \dfrac{\partial L}{\partial r} = mr\dot{\theta}^2 + mg\cos\theta$ $\qquad \dfrac{\partial f}{\partial r} = 1$

$$(1) \quad m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta - \lambda = 0$$

θ: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} = \dfrac{d}{dt}(mr^2\dot{\theta})$ $\qquad \dfrac{\partial L}{\partial \theta} = -mgr\sin\theta$ $\qquad \dfrac{\partial f}{\partial \theta} = 0$

$$(2) \quad \frac{d}{dt}(mr^2\dot{\theta}) + mgr\sin\theta = 0$$

Constraint (3) $r = l$

Solve: immediately have $\dot{r} = \ddot{r} = 0$

So (1) $\rightarrow$ $-ml\dot{\theta}^2 - mg\cos\theta = \lambda$

(2) $\rightarrow$ $ml^2\ddot{\theta} + mgl\sin\theta = 0$

$$\rightarrow \quad \ddot{\theta} + \omega^2 \sin\theta = 0$$

regular pendulum equation ✓

Given solution, have $\lambda = \lambda \frac{\partial f}{\partial r} = Q_r$.

$$\text{So} \quad Q_r = -ml\dot{\theta}^2 - mg\cos\theta$$

If we tried to increase r, would do work

$$\delta W = Q_r\,\delta r = -T\,\delta r$$

tension directed inward

$$\text{So} \quad T = -Q_r = ml\dot{\theta}^2 + mg\cos\theta$$

, as before ✓

Note that we actually can solve for $T(\theta)$:

Know energy is conserved $\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = E \equiv -mgl\cos\theta_0$

$$\text{so} \quad \dot{\theta}^2 = \frac{2g}{l}\left(\cos\theta - \cos\theta_0\right)$$

$\theta_0 =$ max height of pendulum

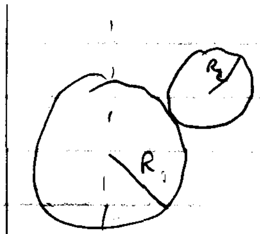
$$\text{So} \quad T = ml\left[\frac{2g}{l}\left(\cos\theta - \cos\theta_0\right)\right] + mg\cos\theta$$

$$\boxed{T = mg\left(3\cos\theta - 2\cos\theta_0\right)}$$

Note for such a simple problem, Lagrange multiplier formalism isn't worth effort.

Look at more complicated one

Set up: Cylinder, radius R_1, fixed in space
Cylinder radius R_2 rolls on fixed cylinder w/o slipping
Calculated motion in gravity g

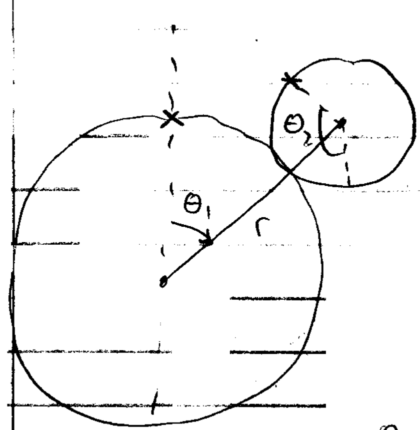


Questions: - When does cylinder fall off?
- What is friction force?

Here two constraints all together:
- Cylinders roll w/o slipping (friction force)
- Separation of centers = $R_1 + R_2$ (normal force)

Use two Lagrange multipliers λ_1, λ_2

Set up coords independent of constraints first

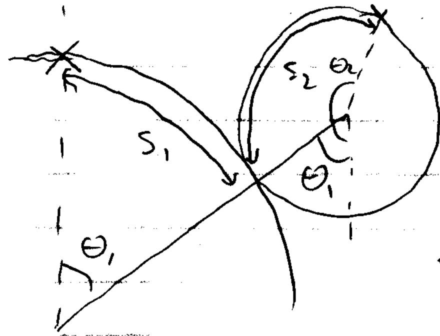


θ_1 = angle of center from vertical

θ_2 = angle cylinder has rotated

r = center separation

Rolling constraint a bit tricky:



Need $s_1 = s_2$

$$R_1\theta_1 = R_2(\theta_2 - \theta_1)$$

$$\rightarrow f_2(\theta_1, \theta_2) = (R_1+R_2)\theta_1 - R_2\theta_2 = 0$$

Other constraint is $f_1(r) = r - R_1 - R_2 = 0$

Set up Lagrangian:

$$T = \underbrace{\tfrac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}_1^2)}_{\text{center of mass}} + \underbrace{\tfrac{1}{2} I \dot{\theta}_2^2}_{\text{rotation about CM}} \qquad V = mgr\cos\theta_1$$

$I = \frac{1}{2} m R_2^2$ moment of inertia

Equations of motion:

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial r} = mr\dot{\theta}_1^2 - mg\cos\theta_1 \qquad \frac{\partial f_1}{\partial r} = 1 \qquad \frac{\partial f_2}{\partial r} = 0$$

So $\boxed{m\ddot{r} - mr\dot{\theta}_1^2 + mg\cos\theta_1 - \lambda_1 = 0}$

and $\frac{\partial L}{\partial \dot{\theta}_1} = mr^2\dot{\theta}_1 \qquad \frac{\partial f_1}{\partial \theta_1} = 0 \qquad \frac{\partial f_2}{\partial \theta_1} = R_1 + R_2$

$$\frac{\partial L}{\partial \theta_1} = mgr\sin\theta_1$$

$$\boxed{\frac{d}{dt}\left[mr^2\dot{\theta}_1\right] - mgr\sin\theta_1 - \lambda_2(R_1+R_2) = 0}$$

and $\frac{\partial L}{\partial \dot{\theta}_2} = I\dot{\theta}_2 \qquad \frac{\partial L}{\partial \theta_2} = 0 \qquad \frac{\partial f_1}{\partial \theta_2} = 0 \qquad \frac{\partial f_2}{\partial \theta_2} = -R_2$

$$\boxed{I\ddot{\theta}_2 + R_2\lambda_2 = 0}$$

and $\boxed{r = R_1 + R_2}$

$$\boxed{\theta_2 = \left(1 + \frac{R_1}{R_2}\right)\theta_1}$$

Solve, in terms of θ_1:

$$\boxed{-m(R_1+R_2)\dot{\theta}_1^2 + mg\cos\theta_1 = \lambda_1}$$

$$\frac{1}{2}mR_2^2\left(1+\frac{R_1}{R_2}\right)\ddot{\theta}_1 + R_2\lambda_2 = 0$$

$$\Rightarrow \boxed{\lambda_2 = -\frac{1}{2}m(R_2+R_1)\ddot{\theta}_1}$$

and

$$m(R_1+R_2)^2\ddot{\theta}_1 - mg(R_1+R_2)\sin\theta_1 + \frac{1}{2}m(R_1+R_2)^2\ddot{\theta}_1 = 0$$

$$\boxed{\ddot{\theta}_1 - \frac{2g}{3(R_1+R_2)}\sin\theta_1 = 0}$$

pendulum again

Solve using same trick

mathematically, multiply by $\dot{\theta}_1$

$$\dot{\theta}_1\ddot{\theta}_1 - \frac{2g}{3(R_1+R_2)}\dot{\theta}_1\sin\theta = 0$$

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}_1^2 + \frac{2g}{3(R_1+R_2)}\cos\theta\right] = 0$$

$$\dot{\theta}_1^2 + \frac{4g}{3(R_1+R_2)}\cos\theta = \text{const}$$

Get const from initial condition $\theta(0)=0$, $\dot{\theta}(0)=0$

$$\dot{\theta}_1^2 = \frac{4g}{3(R_1+R_2)}(1-\cos\theta_1)$$

So $\lambda_1 = -\frac{4}{3}mg(1-\cos\theta_1) + mg\cos\theta_1$

$$= mg\left(-\frac{4}{3}+\frac{7}{3}\cos\theta_1\right)$$

For first question, want to know when $\lambda \to 0$

at $\boxed{\cos\Theta_1 = \frac{4}{7}}$ $\Theta_1 = 55.15^\circ$

independent of R_1, R_2!

For second question, get $\lambda_2 = -\frac{1}{2} m (R_1 + R_2)\ddot{\Theta}_1$

$$= -\frac{1}{2} m (R_1 + R_2)\left[\frac{2g}{3(R_1+R_2)} \sin\Theta_1\right]$$

$$\lambda_2 = -\frac{1}{3} mg \sin\Theta_1$$

Generalized force $Q_{\Theta_1} = \frac{\partial f_2}{\partial \Theta_1} \lambda_2$

$$= -\frac{1}{3}(R_1 + R_2)\, mg \sin\Theta_1$$

Get cartesian force from friction:

Try to increase Θ_1 without rolling
do work $Q_{\Theta_1}\delta\Theta_1 = F_f \times R_1 \delta\Theta_1$

$$\boxed{F_f = -\frac{1}{3}\frac{R_1 + R_2}{R_1} mg \sin\Theta_1}$$

Could do this problem with more elementary methods, but probably harder. (If you remember how Lagrange multiplier technique works.)