Lecture 5 - Lagrangian Techniques

Last time, developed Lagrangian formalism

$$L = T - V \quad = \text{function of generalized coordinates } q_\sigma$$

then $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \qquad \text{for each } \sigma$$

Provides equations of motion without explicitly including constraint forces

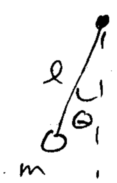
Often convenient to use polar (r, Θ) or spherical (r, ϑ, ϕ) coords

For reference: $$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\Theta}^2\right) \qquad \text{polar}$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\vartheta}^2 + r^2 \dot{\Theta}^2 \sin^2\vartheta\right) \qquad \text{spherical}$$

not bad things to memorize

Example: pendulum



$r = l$ fixed, so $$T = \frac{1}{2} m l^2 \dot{\Theta}^2$$

$$V = -mgl\cos\Theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\Theta}} = m l^2 \ddot{\Theta} \qquad \frac{\partial L}{\partial \Theta} = -\frac{\partial V}{\partial \Theta} = -mgl\sin\Theta$$

Equation of motion $$\ddot{\Theta} + \omega^2 \sin\Theta = 0 \qquad \omega^2 = \frac{g}{l}$$

Even such a simple system has no general analytic solution

If real solution needed, solve numerically

But can get analytic results for motion near equilibrium

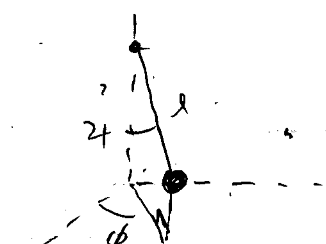
Here equilibrium solution $\phi = 0$

Suppose ϕ close to zero:
then $\sin\phi \approx \phi$, get

$$\ddot{\phi} + \omega^2 \phi = 0$$

Simple harmonic oscillator, soluable

More complex example: space pendulum



Here $T = \frac{1}{2} m l^2 (\dot{\psi}^2 + \dot{\phi}^2 \sin^2\psi)$

$V = -mgl\cos\psi$

$$\frac{\partial L}{\partial \dot{\psi}} = m l^2 \dot{\psi} \qquad \frac{\partial L}{\partial \psi} = m l^2 \dot{\phi}^2 \sin\psi\cos\psi - mgl\sin\psi$$

So $$\ddot{\psi} - \dot{\phi}^2 \sin\psi\cos\psi + \omega_0^2 \sin\psi = 0$$

$\omega_0^2 = \frac{g}{l}$

$$\frac{\partial L}{\partial \dot{\phi}} = m l^2 \dot{\phi} \sin^2\psi \qquad \frac{\partial L}{\partial \phi} = 0$$

So $$\frac{d}{dt}(\dot{\phi}\sin^2\psi) = 0$$

$$\dot{\phi}\sin^2\psi = b = \text{const}$$

Again, can't solve. But can find equilibrium $\ddot{\theta} = 0$:

use $\dot{\phi} = \dfrac{b}{\sin^2\theta}$

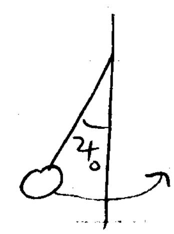
$$-\left(\frac{b}{\sin^2\theta}\right)^2 \sin\theta\cos\theta + \omega_0^2 \sin\theta = 0$$

Either $\sin\theta = 0$ (boring) or $b^2\cos\theta = \omega_0^2 \sin^4\theta$ (interesting)

Clear that set. solution $\theta = \theta_0$ const

Mass executes uniform circular motion

Rotates at $\dot{\phi} = \dfrac{b}{\sin^2\theta_0} = \Omega$ also constant



Can solve for θ_0 in terms of Ω:

$$-\Omega^2 \sin\theta_0\cos\theta_0 + \omega_0^2 \sin\theta_0 = 0$$

$$\cos\theta_0 = \frac{\omega_0^2}{\Omega^2} = \frac{g}{l\,\Omega^2}$$

Now suppose motion near equilibrium:
Expand equations for $\theta = \theta_0 + \delta$

Have $\dot{\phi}\sin^2\theta = b = \Omega\sin^2\theta_0$

So $$\ddot{\theta} - \left(\Omega^2 \frac{\sin^4\theta_0}{\sin^4\theta}\right)\sin\theta\cos\theta + \omega_0^2\sin\theta = 0$$

$\theta = \theta_0 + \delta$, expand to first order:

$$\sin\psi \to \sin\psi_0 + \delta\cos\psi_0$$

$$\cos\psi \to \cos\psi_0 - \delta\sin\psi_0$$

$$\frac{1}{\sin^3\psi} \to \frac{1}{\sin^3\psi_0}\left(1 - 3\delta\,\frac{\cos\psi_0}{\sin\psi_0}\right)$$

Get

$$\ddot{\delta} - \Omega^2\left(1 - 3\delta\,\frac{\cos\psi_0}{\sin\psi_0}\right)\sin\psi_0\,(\cos\psi_0 - \delta\sin\psi_0)$$

$$+\,\omega_0^2(\sin\psi_0 + \delta\cos\psi_0) = 0$$

Zero order terms:

$$-\Omega^2\sin\psi_0\cos\psi_0 + \omega_0^2\sin\psi_0 = 0$$

$$\Rightarrow \cos\psi_0 = \frac{\omega_0^2}{\Omega^2} \quad \checkmark$$

First order:

$$\ddot{\delta} + 3\Omega^2\delta\cos^2\psi_0 + \Omega^2\delta\sin^2\psi_0 + \omega_0^2\delta\cos\psi_0 = 0$$

$$\ddot{\delta} + 2\Omega^2\cos^2\psi_0\,\delta + \Omega^2\delta + \omega_0^2\delta\cos\psi_0 = 0$$

$$\ddot{\delta} + 2\Omega^2\frac{\omega_0^4}{\Omega^4}\delta + \Omega^2\delta + \omega_0^2\delta\,\frac{\omega_0^2}{\Omega^2} = 0$$

$$\ddot{\delta} + \left(3\frac{\omega_0^4}{\Omega^2} + \Omega^2\right)\delta = 0$$

Harmonic oscillator equation

$$\text{Freq} \quad \omega^2 = 3\frac{\omega_0^4}{\Omega^2} + \Omega^2 = \omega_0^2\left(3\cos\psi_0 + \frac{1}{\cos\psi_0}\right)$$

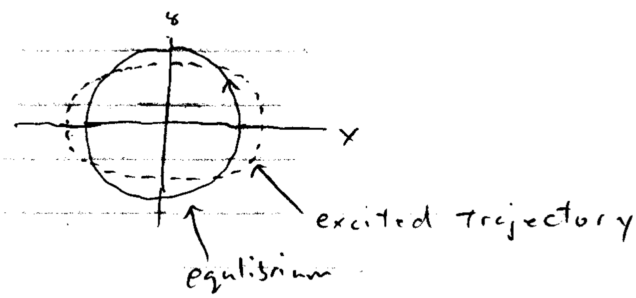
Mass bobbles at this frequency as it swings around

Interesting point:

For $\psi_0 \to 0$, $\omega \to 2\omega_0 = 2\sqrt{\frac{g}{\ell}}$ ... surprising

Do get $\omega = \sqrt{\frac{g}{\ell}}$ if we take $\dot{\phi} = 0$ from beginning

But here still have small 3D amplitude:



See that deviation oscillates 2x faster than rotation rate

You get to do several problem like this in HW

Now, say a bit more about constraint forces (tension, normal force, etc)

Virtue of Lagrangian method is that we don't worry about them.

But what if we want to know?

Example: how fast can we swing pendulum before string breaks?

Often, this is easy to answer by elementary methods:

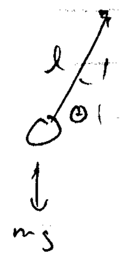
Know $q_\sigma(t)$ - generalized coords

Use to get $x_i(t)$ - Cartesian coords

Get $m_i \ddot{x}_i = F_i$ = force

Knowing total force, usually can identify portion from constraints

Example: pendulum



Given $\theta(t)$, know

centripetal $a_c = \ell \dot{\theta}^2$
tangential $a_t = \ell \ddot{\theta}$

$$m a_c = T - mg \cos\theta$$
$$m a_t = mg \sin\theta$$

So $T = m \ell \dot{\theta}^2 + mg \cos\theta$

Easy enough

But in complicated problem, untangling a particular constraint force can be hard

Have a general approach:

Hamilton's principle tells us

$$\int_{t_1}^{t_2} \left[\sum_\sigma \delta q_\sigma \left(\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \right) \right] dt = 0$$

Suppose we ignored one (or more) constraints when setting up coords

For instance, kept r $(=\ell)$ as free coord in pendulum

Then we also have constraints $f(q, t) = 0$

For instance, $r - \ell = 0$

Then we need to minimize action, subject to constraints

Do using Lagrange multipliers

Recall from calculus: minimize $\phi(x,y)$ subject to $f(x,y)=0$

Make new function $\mathcal{L}(x,y,\lambda) = \phi - \lambda f$

Minimize $\mathcal{L}$ w/respect to x, y, and λ

Gives constrained minimum of ϕ

Don't see proof very often... I'll post one on web page.

Here generalize to functionals:

Have $f(q_\sigma)=0$

If we vary path $q+\delta q$ such that constraint still satisfied, need $\delta f = \sum_\sigma \frac{\partial f}{\partial q_\sigma} \delta q_\sigma = 0$

Multiply by arbitrary functions $\lambda(q,\dot{q},t)$, still zero

Add to integrand; still zero:

$$\int \sum_\sigma \delta q_\sigma \left[\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} \right] dt = 0$$

one term for each additional constraint

If we have n coords and k constraints, really only $n-k$ degrees of freedom

$\Rightarrow$ different δq_σ's are not independent

So we can demand that [ ] terms vanish for $\sigma = 1$ to $n-k$ only

But we can now choose λ_k's such that [ ] for $\sigma = n-k+1$ to n vanish as well

End up with

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} - \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0 \qquad \sigma = 1 \text{ to } n$$

and $\qquad f_j(q) = 0 \qquad\qquad j = 1$ to k

Total of $n+k$ equations

Have $n+k$ unknowns: $\{q_\sigma\}$ and $\{\lambda_j\}$

So we can solve for q's & λ's: gives constrained solution

Why do all this?

Can see physical significance of λ:

Suppose we instead included constraint forces directly in L

Then would have constraint potential V_c in

$$L \to T - V - V_c = L^{(0)} - V_c(q's)$$

Get eqution of motion

$$\frac{d}{dt}\frac{\partial L^{(0)}}{\partial \dot{q}_\sigma} - \frac{\partial L^{(0)}}{\partial q_\sigma} + \frac{\partial V_c}{\partial q_\sigma} = 0$$

See that term $\lambda \frac{\partial f}{\partial q_\sigma}$ corresponds to $-\frac{\partial V_c}{\partial q_\sigma}$

= "generalized" constraint force Q_σ

or, $$\boxed{\lambda_j \frac{\partial f_j}{\partial q_\sigma} \equiv Q_\sigma^{(j)}}$$

So, knowing λ lets you get constraint forces

Can relate to proper force F_σ by work:
If motion δq_σ violates constraint, then constraint force does work

$$\delta W = Q_\sigma \delta q_\sigma = \sum_i F_i \delta x_i$$

Given $x_i(q_\sigma\text{'s})$, can solve for F_i's as desired

Usually straightforward

Example: pendulum

Set up with r, θ both free

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta$$

Constraint $\quad f_1(r,\theta) = r - l = 0$

$$\frac{\partial f}{\partial r} = 1 \qquad \frac{\partial f}{\partial \theta} = 0$$

Get equations of motion:

(1) $m(\ddot{r} - r\dot{\theta}^2) - mg\cos\theta = \lambda$

(2) $\frac{d}{dt}(mr^2\dot{\theta}) + mgr\sin\theta = 0$

(3) $r = l$

In solution, immediately have $\dot{r} = \ddot{r} = 0$

(1) $\rightarrow$ $-ml\dot{\theta}^2 - mg\cos\theta = \lambda$

and

(2) $\rightarrow$ $\underbrace{ml^2\ddot{\theta} + mgl\sin\theta = 0}$

$\hookrightarrow \ddot{\theta} + \omega^2\sin\theta = 0$
as before

Given solution, have $\lambda = \lambda\frac{\partial f}{\partial r} = Q_r$

so $Q_r = -ml\dot{\theta}^2 - mg\cos\theta$

If we tried to increase r, would do work

$$\delta W = Q_r\,\delta r = -T\,\delta r$$

tension directed inward, but δr outward

So: tension $T = -Q_r$
$= ml\dot{\theta}^2 + mg\cos\theta$

As before. Such a simple problem doesn't merit complicated formalism, but we'll see more complicated example next time