Lecture 4 - Lagrangians

Last time, discussed classical scattering theory

Key parameter $d\sigma/d\Omega$ relates theory & experiment

Some concepts apply to quantum scattering

Still get $d\sigma/d\Omega$ from experiment

but quantum calculation is different

That wraps up Ch 1 ... jump to Ch 3

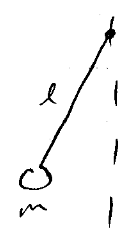
Lagrangian Mechanics

= reformulation of Newton's Laws

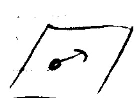
Why? Convenient for constrained systems

Some examples: Simple pendulum

Constraint = distance between mass and support = l



Particle on plane



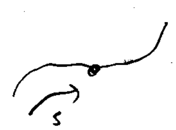
Constraint: Plane normal $\hat{n}$,

need $\vec{r} \cdot \hat{n} = 0$

Bead sliding on wire

Constraint $\vec{r} = \vec{r}(s)$

wire parametrized by s



Constrained problems very common

Always imply "constraint forces" : forces that ensure constraint satisfied

Tension force, normal force

In Newton formulation, need to include constraint forces & solve for them

Lagrangian handles automatically : very powerful

Set up:

N particles, $n = 3N$ coordinates $x_1, \ldots x_n$

k constraints: functions $f_i(x_1, \ldots x_n, t) = 0$ $\quad i = 1$ to k

Leaves $n-k$ degrees of freedom

Represent with parameters $q_1, \ldots q_{n-k}$

= any set of values that (along with f's) determine x's

So $x_i = x_i(q_1, \ldots q_{n-k}, t)$ $\quad i = 1$ to n

Flexible choice of q's : will give examples of good choices

Want to transform $F_i = m\ddot{x}_i$ from x's to q's

Book goes through, sections 13-15
- Interesting but not trivial
- To make room at end, I'm going to skip
- Instead, argue other way:

Define $L(\{q_\sigma, \dot{q}_\sigma\}) = T(\{q_\sigma, \dot{q}_\sigma\}) - V(\{q_\sigma\})$

L = Lagrangian

T = kinetic energy expressed as function of q's

V = potential energy

Get T from relations $x(q)$:

$$T = \sum_i \frac{1}{2} m_i \dot{x}_i^2 \qquad m_i = \text{mass associated with } x_i$$

$$\dot{x}_i = \frac{d}{dt} x_i(q_1, \dots q_{n-k}, t)$$

$$= \frac{\partial x_i}{\partial q_1}\dot{q}_1 + \frac{\partial x_i}{\partial q_2}\dot{q}_2 + \dots + \frac{\partial x_i}{\partial q_{n-k}}\dot{q}_{n-k} + \frac{\partial x_i}{\partial t}$$

Recall $\frac{\partial x_i}{\partial q_1}$ means derivative w/respect to q_1, all other parameters held constant

(Sometimes some short cuts, will see in examples later)

Now for a given trajectory $q_\sigma = q_\sigma(t)$ define action

$$S = \int_{t_1}^{t_2} L(q_\sigma)\, dt$$

t_1 = starting time of trajectory

t_2 = ending time

Then can state Hamilton's Principle:

> When a mechanical system evolves from $q_\sigma^{(1)}(t_1)$ to $q_\sigma^{(2)}(t_2)$, it's motion is such that S is a minimum.

To use this, need to know calculus of variations

Math problem:

Given relation $\phi(y, y', x)$ $\quad\left(y' = \frac{dy}{dx}\right)$

Find function $y(x)$ such that:

a) $y(x_1) = y_1$

b) $y(x_2) = y_2$

c) $I[y] = \int_{x_1}^{x_2} \phi \, dx$ is an extremum (large or small as possible)

Call $I[y]$ a functional:

<u>Function</u> $f(x)$: takes number, returns number

Functional $F[y]$: takes function, returns number

Operator $\mathcal{U}y$: takes function, returns function

(If you take a number and return a function, think of number as a <u>parameter</u>.)

So we need to find extremes of a functional

Like setting $\frac{df}{dx} = 0$

To solve, suppose we knew correct $y(x)$

Consider slightly different $Y(x) = y(x) + \delta y(x)$

δy is small, and $\delta y(x_1) = \delta y(x_2) = 0$

Then $I[Y] = \int_{x_1}^{x_2} \phi(Y, Y', x)\, dx$

$$= \int \phi(y + \delta y, y' + \underbrace{\delta y'}_{\frac{d}{dx}\delta y}, x)\, dx$$

Taylor expand:

$$I[Y] \approx \int \left[\phi(y, y', x) + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial y'}\delta y'\right] dx$$

$$= I[y] + \int_{x_1}^{x_2} \left[\frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial y'}\delta y'\right] dx$$

Want in terms of δy, so integrate $\delta y'$ by parts!

$$\int_{x_1}^{x_2} \frac{\partial \phi}{\partial y'} \frac{d}{dx}(\delta y)\, dx = \underbrace{\frac{\partial \phi}{\partial y'}\delta y \Big|_{x_1}^{x_2}}_{=0} - \int_{x_1}^{x_2} \delta y \frac{d}{dx}\left(\frac{\partial \phi}{\partial y'}\right) dx$$

So $$I[Y] \approx I[y] + \int_{x_1}^{x_2} \left[\frac{\partial \phi}{\partial y} - \frac{d}{dx}\frac{\partial \phi}{\partial y'}\right] \delta y\, dx$$

Now if $I[y]$ is extremum, either

- All nearby Y's have $I[Y] < I[y]$ max
- " " " $I[Y] > I[y]$ min
- " " " $I[Y] = I[y]$ flat

But if $I[Y]$ has term linear in δy, none of these hold:

If $I[y+\delta y] > I[y]$, then $I[y-\delta y] < I[y]$

and $I[Y] \neq I[y]$ since difference is 1st order

So if $I[y]$ is really extreumm, must have

$$\int_{x_1}^{x_2} \left[\frac{\partial \phi}{\partial y} - \frac{d}{dx} \frac{\partial \phi}{\partial y'} \right] \delta y \; dx = 0$$

for all δy Requires $\boxed{\frac{\partial \phi}{\partial y} - \frac{d}{dx} \frac{\partial \phi}{\partial y'} = 0}$

= Euler's equation

= Diff eqn we can solve for $y(x)$

Example: shortest distance between two points (x_1, y_1) and (x_2, y_2)

$$I = \int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int \sqrt{1+y'^2} \, dx$$

$$\phi = \sqrt{1+y'^2}$$

$$\frac{\partial \phi}{\partial y} = 0 \qquad \frac{\partial \phi}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

So $\frac{d}{dx} \frac{y'}{\sqrt{1+y'^2}} = 0 \Rightarrow \frac{y'}{\sqrt{1+y'}} = \text{const}$

$\Rightarrow y' = \text{const}$

$\Rightarrow y = mx + b$, straight line

Call $I[y+\delta y] - I[y] = \delta I$ "variation in I"
so need $\delta I = 0$

Variational techiques very useful
Optics: Fermat's principle
QM: variational principle
General Relativity: geodesics

Mechanics

$$S = S[q_\sigma] = \int_{t_1}^{t_2} L\, dt$$

Hamilton's principle says $\delta S = 0$ (for each σ)

Euler's equation demands $\dfrac{\partial L}{\partial q_\sigma} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_\sigma} = 0$ $\sigma = 1$ to $n-k$

usually write $\boxed{\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_\sigma} - \dfrac{\partial L}{\partial q_\sigma} = 0}$

call it Lagrange's equation here.

Easy to see this is consistent with Newton:
take $q_\sigma = x_\sigma$

Then $L = \sum_\sigma \frac{1}{2} m_\sigma \dot{x}_\sigma^2 - U(x_1, \dots x_n)$

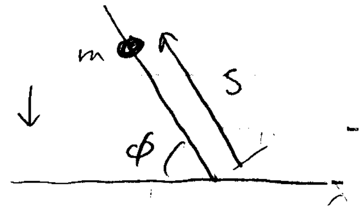
$$\frac{\partial L}{\partial \dot{q}_\sigma} = m_\sigma \dot{x}_\sigma \qquad \frac{\partial L}{\partial q_\sigma} = -\frac{\partial U}{\partial x_\sigma}$$

$$m_\sigma \ddot{x}_\sigma + \frac{\partial U}{\partial x_\sigma} = 0$$

$$m_\sigma \ddot{x}_\sigma = -\frac{\partial U}{\partial x_\sigma} = F_\sigma$$

Conclude motion given by Newton satisfies Hamilton
Statement independent of coords, so fine
if you use q's instead

Examples



Mass on wire
wire fixed, angle ϕ

Coord s = distance along wire

$$T = \frac{1}{2} m \dot{s}^2 \qquad V = mgs \sin\phi$$

$$L = \frac{1}{2} m \dot{s}^2 - mgs \sin\phi$$

$$\frac{\partial L}{\partial \dot{s}} = m\dot{s} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = m\ddot{s} \qquad \frac{\partial L}{\partial s} = -mg \sin\phi$$

so
$$m\ddot{s} - mg\sin\phi = 0$$
$$\boxed{\ddot{s} = g \sin\phi}$$
easy

More complicated:
wire free to pivot (but massless)

Then $\phi \to$ coord

Be carefully getting T:

$$x = s\cos\phi \qquad \dot{x} = \dot{s}\cos\phi - s\dot{\phi}\sin\phi$$
$$y = s\sin\phi \qquad \dot{y} = \dot{s}\sin\phi + s\dot{\phi}\cos\phi$$

$$\dot{x}^2 + \dot{y}^2 = \dot{s}^2\cos^2\phi - 2\dot{s}s\dot{\phi}\cos\phi\sin\phi + s^2\dot{\phi}^2\sin^2\phi$$
$$+ \dot{s}^2\sin^2\phi + 2\dot{s}s\dot{\phi}\cos\phi\sin\phi + s^2\dot{\phi}^2\cos^2\phi$$

$$= \dot{s}^2 + s^2\dot{\phi}^2$$

Handy short cut: if each coordinate corresponds to an orthogonal motion, then compute T's independently

Here motion from s $\perp$ motion from ϕ

$\Rightarrow T = T_s + T_\phi$

not true in general!

Here get $L = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}ms^2\dot{\phi}^2 - mgs\sin\phi$

$$\frac{\partial L}{\partial \dot{s}} = m\dot{s} \qquad \frac{\partial L}{\partial s} = ms\dot{\phi}^2 - mg\sin\phi$$

So $\boxed{m\ddot{s} - ms\dot{\phi}^2 + mg\sin\phi = 0}$

also $\frac{\partial L}{\partial \dot{\phi}} = ms^2\dot{\phi} \qquad \frac{\partial L}{\partial \phi} = -mgs\cos\phi$

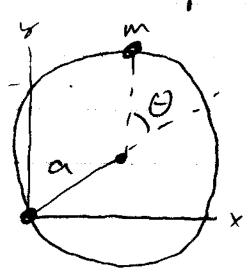
$$\boxed{\frac{d}{dt}(ms^2\dot{\phi}) + mgs\cos\phi = 0}$$

Two equations, two unknowns

Generally, hard / impossible to solve analytically

For now work on setting up

Another example: bead on rotating hoop



hoop rotates about origin point at rate ω

wire slides on hoop

Use coordinate θ as shown

Then $x = a\cos\omega t + a\cos(\omega t + \theta)$

$y = a\sin\omega t + a\sin(\omega t + \theta)$

$$\dot{x} = -\omega a\sin\omega t - a(\omega + \dot{\theta})\sin(\omega t + \theta)$$
$$\dot{y} = \omega a\cos\omega t + a(\omega + \dot{\theta})\cos(\omega t + \theta)$$

$$\dot{x}^2+\dot{y}^2 = a^2\omega^2 \sin^2\omega t + a^2(\omega+\dot\theta)^2\sin^2(\omega t+\theta) + 2a^2\omega^2(\omega+\dot\theta)\sin\omega t\,\sin(\omega t+\theta)$$

$$+ a^2\omega^2\cos^2\omega t + a^2(\omega+\dot\theta)^2\cos^2(\omega t+\theta) + 2a^2\omega(\omega+\dot\theta)\cos\omega t\,\cos(\omega t+\theta)$$

$$= a^2\omega^2 + a^2(\omega+\dot\theta)^2 + 2a^2\omega(\omega+\dot\theta)\left[\sin\omega t\,\sin(\omega t+\theta) + \cos\omega t\,\cos(\omega t+\theta)\right]$$

$$= a^2\left[\omega^2 + (\omega+\dot\theta)^2 + 2\omega(\omega+\dot\theta)\cos\theta\right]$$

$$L = T = \frac{1}{2}ma^2\left[\omega^2 + (\omega+\dot\theta)^2 + 2\omega(\omega+\dot\theta)\cos\theta\right]$$

since $U=0$

Note t drops out

$$\frac{\partial L}{\partial\dot\theta} = ma^2(\omega+\dot\theta) + ma^2\omega\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = ma^2\ddot\theta - ma^2\omega\dot\theta\sin\theta$$

$$\frac{\partial L}{\partial\theta} = -ma^2\omega(\omega+\dot\theta)\sin\theta$$

so $ma^2\ddot\theta - ma^2\omega\dot\theta\sin\theta + ma^2\omega(\omega+\dot\theta)\sin\theta = 0$

$$\boxed{\ddot\theta + \omega^2\sin\theta = 0}$$

Note, this is same as equation for pendulum

Next time: discuss how to find solutions at / near equilibrium

Guest lecture: