Lecture 3 - Scattering

Last time, considered central forces, gravity

$U(r) = -\frac{\gamma\mu}{r}$ solution $\frac{1}{r} = C(1-\varepsilon\cos\phi)$

$C = \frac{\mu^2\gamma}{\ell^2}$ $\varepsilon^2 = 1 + \frac{2E\ell^2}{\mu^3\gamma^2}$

μ = mass E = energy ℓ = ang. momentum

If $E<0$, $0<\varepsilon<1$, ellipse

Today consider $E>0$

particle unbound, $r \to \infty$

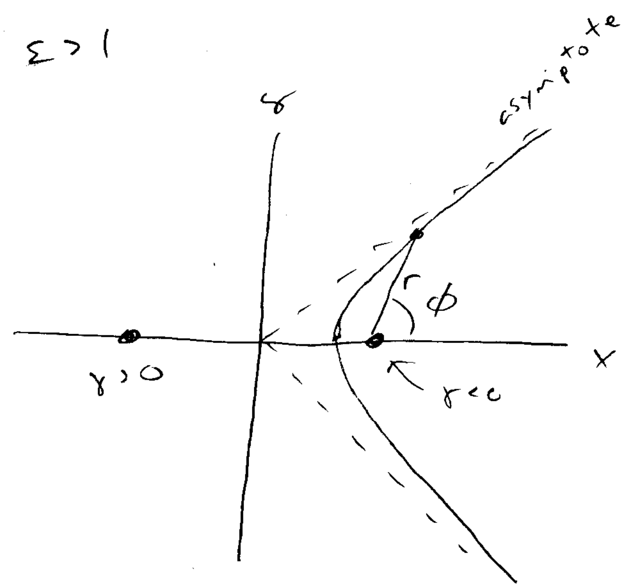
Could have $\gamma > 0$ (repulsive)

or $\gamma < 0$ (attractive)

Orbit still valid, now $\varepsilon > 1$

$\to$ orbit = hyperbola

(parabola for $\varepsilon = 1$)



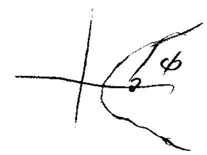
Center at one focus:

If $\gamma>0$, $C>0$

Need $r>0$ always, so $1-\varepsilon\cos\phi > 0$

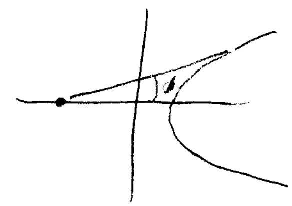
Need to avoid $\cos\phi > \frac{1}{\varepsilon}$

avoid $\phi = 0$



If $\gamma < 0$, $C < 0$, need $1 - \varepsilon\cos\phi < 0$

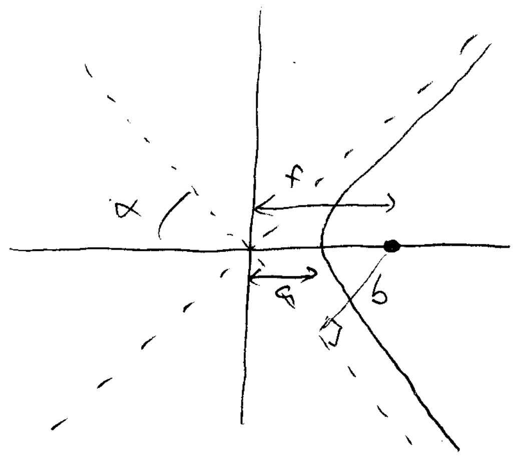
avoid $\phi = \pi$



Hyperbola geometry

Still have $a = \left|\frac{\mu\gamma}{2E}\right|$

$f = \varepsilon a$



Closest approach

$= f - a \qquad (\gamma > 0)$

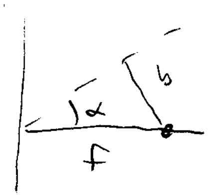
or $\quad f + a \qquad (\gamma < 0)$

Angle of asymptote $= \alpha$

$= \phi(r \to \infty)$

$0 = 1 - \varepsilon\cos\alpha \quad \to \quad \boxed{\cos\alpha = \frac{1}{\varepsilon}}$

Ellipse had $\quad b = \sqrt{a^2 - f^2}$



Here $\quad b = f\sin\alpha$

$= f\sqrt{1 - \cos^2\alpha}$

$= \sqrt{f^2 - f^2/\varepsilon^2}$

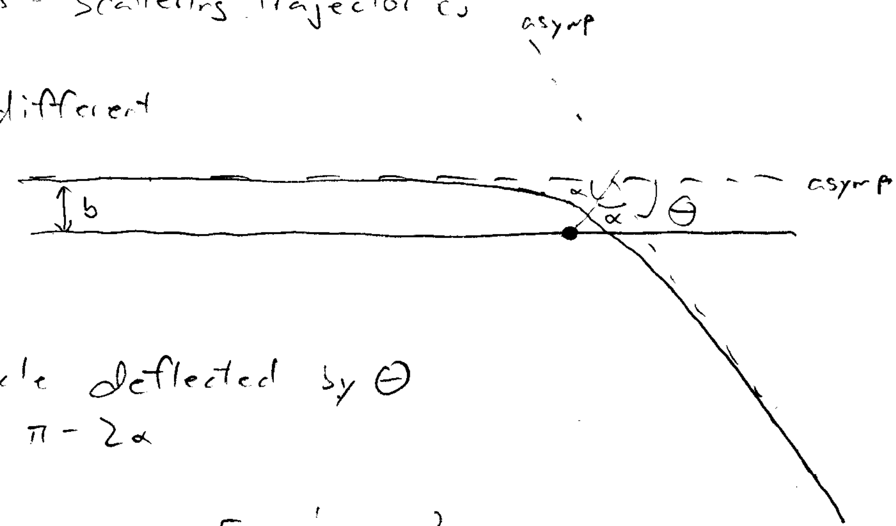
$= \sqrt{f^2 - a^2} \quad = a\sqrt{\varepsilon^2 - 1}$

Gives distance from asymptote to center

Also called "impact parameter"

Unbound orbits = scattering trajectories

Draw picture different



Incident particle deflected by Θ

$$\Theta = \pi - 2\alpha$$

Write particle energy $E = \frac{1}{2}\mu v_\infty^2$

See $\ell = \mu b v_\infty$

Get Θ in terms of b, v_∞:

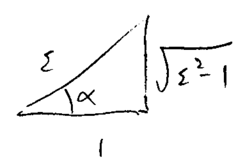
Start with $\varepsilon^2 = 1 + \dfrac{2E\ell^2}{\mu^3\gamma^2}$

$$\varepsilon^2 = 1 + \frac{(\mu v_\infty^2)(\mu b v_\infty)^2}{\mu^3\gamma^2} = 1 + \frac{v_\infty^4 b^2}{\gamma^2}$$

Then $\cos\alpha = \dfrac{1}{\varepsilon}$ and $\dfrac{\Theta}{2} = \dfrac{\pi}{2} - \alpha$

$$\Rightarrow \cot\frac{\Theta}{2} = \tan\alpha$$

$$\tan\alpha = \sqrt{\varepsilon^2 - 1}$$



$$= \frac{v_\infty^2 b}{\gamma}$$

and $\boxed{\cot\dfrac{\Theta}{2} = \dfrac{v_\infty^2 b}{\gamma}}$

Given γ, predict Θ

Or, measure Θ to determine γ $\Leftarrow$

- Basic idea of collision experiments:
  - Measure Θ for various v_∞, b
  - Use to construct unknown $U(r)$

- Main difficulty: cannot control b

  Send in beam of particles, spread over area

  Measure prob to scatter at angle Θ

  Handle averaging over b with <u>cross section</u>

- Define flux $F = \dfrac{\#\ \text{incident particles}}{(\text{area})(\text{time})}$ $\left(\dfrac{1}{cm^2\, s}\right)$

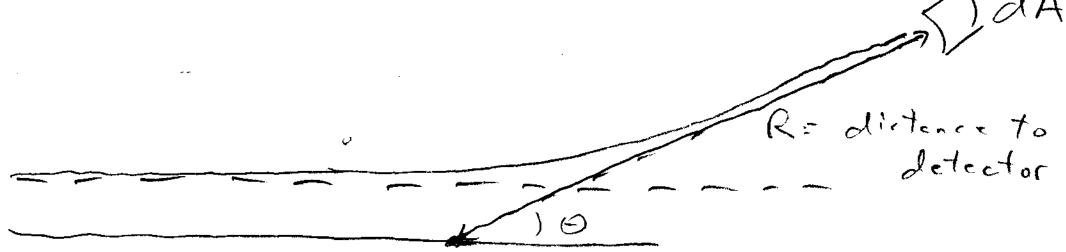
- Measure $dR(\Theta) = \dfrac{\#\ \text{particles detected at } \Theta}{(\text{time})}$ $\left(\dfrac{1}{s}\right)$

- Define $d\sigma = \dfrac{dR}{F}$ (cm^2)

  Note dR is differential because it depends on range of angles $d\Theta$ accepted

  Generally, $dR \propto d\Omega$

  $d\Omega$ = solid angle of detector



$$d\Omega \equiv \frac{dA}{R^2}$$

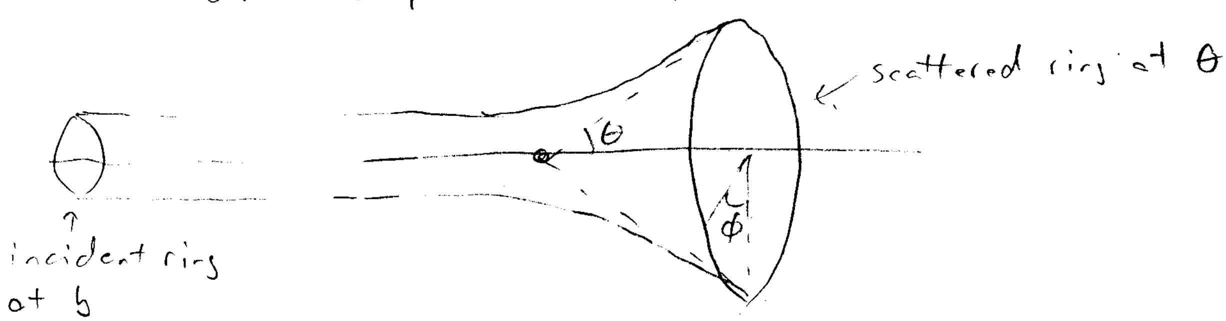
So, expect $d\sigma \propto d\Omega$

$$d\sigma = f(\theta)\, d\Omega$$

$$f(\theta) = \frac{d\sigma}{d\Omega} = \text{differential cross section } (cm^2)$$

Depends only on scattering, not on incident beam or detector

When $U(r)$ spherically symmetric
$d\sigma$ independent of azimuth ϕ



Generally $d\Omega = \sin\theta\, d\theta\, d\phi$

integrate over ϕ: $d\Omega = 2\pi \sin\theta\, d\theta$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi \sin\theta} \frac{d\sigma}{d\theta}$$

Use relation $b = b(\theta)$ from trajectory

Incident ring has width db
$\rightarrow$ area $2\pi b\, db$
Rate particles $= F \times 2\pi b\, db$

All scatter into θ, so detect rate

$$dR = F d\sigma = F\, 2\pi b\, db \Rightarrow d\sigma = 2\pi b\, db$$

Then $\frac{d\sigma}{d\theta} = 2\pi b \left|\frac{db}{d\theta}\right|$ (in case $\frac{db}{d\theta} < 0$)

Thus $$\boxed{\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|}$$

What we measure in experiment

What we calculate from $U(r)$

For $U(r) = -\frac{\gamma\mu}{r}$, $b = \frac{\gamma}{v_\infty^2}\cot\frac{\theta}{2}$

$$\frac{db}{d\theta} = \frac{\gamma}{v_\infty^2}\frac{1}{2}\csc^2\frac{\theta}{2}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{\sin\theta}\left(\frac{\gamma}{v_\infty^2}\cot\frac{\theta}{2}\right)\left(\frac{\gamma}{v_\infty^2}\frac{1}{2}\csc^2\frac{\theta}{2}\right)$$

$$2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \qquad \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \qquad \frac{1}{\sin^2\frac{\theta}{2}}$$

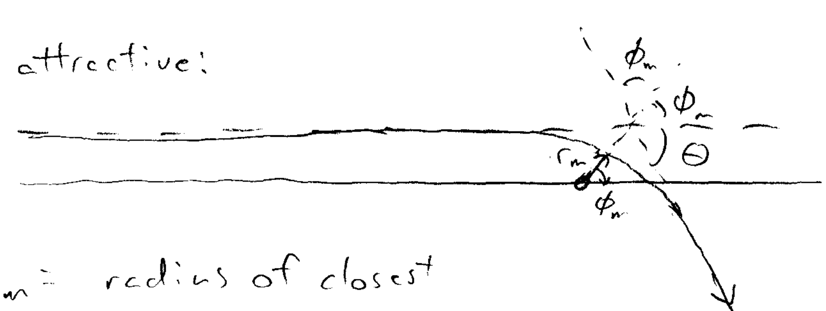
$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\gamma^2}{4v_\infty^2}\frac{1}{\sin^4\frac{\theta}{2}}}$$

Rutherford scattering formula holds for either sign of γ

Even accurate for quantum scattering!

Of course, formalism works for any potential

Say $U(r)$ attractive:



Define r_m = radius of closest approach

Easy to get from $V_{eff}(r_m) = E = \frac{1}{2}\mu v_\infty^2$

Define ϕ_m = polar angle where $r = r_m$

Then $\Theta = \pi - 2\phi_m$

Get ϕ_m from general formula

$$\phi_m = \frac{\ell}{\sqrt{2\mu}} \int_\infty^{r_m} \frac{dr}{r^2\sqrt{E - V_{eff}}} + \pi$$

r from ∞ to r_m

$\phi(r = \infty)$

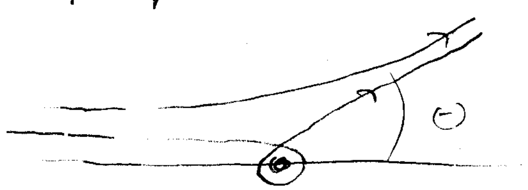
Then $$\boxed{\Theta = \frac{\ell}{2\sqrt{\mu}} \int_{r_m}^\infty \frac{dr}{r^2\sqrt{E - V_{eff}}} - \pi}$$

??

For repulsive potential, $\Theta = 2\phi_m\ \pi$ instead

If both attractive + repulsive, can be complicated

Can get multiple paths with same Θ



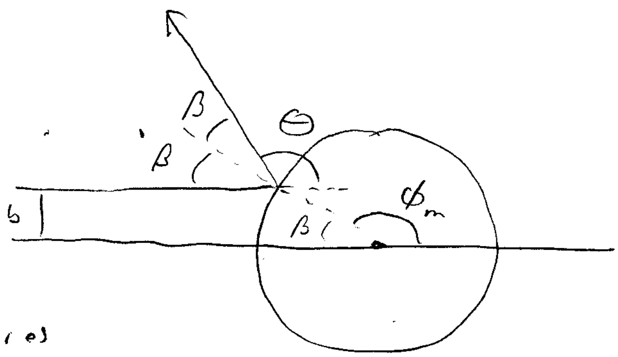
Need to add contributions

Note, in simple problems usually not efficient to use integral formula

Example: hard-sphere scattering $\quad U(r) = \begin{cases} 0 & r > R \\ \infty & r < R \end{cases}$

Integral is tricky

But intuition is clear:



Conservation of l requires
$\beta_{in} = \beta_{out}$

Geometry gives $\quad \sin\phi_m = \frac{b}{R}$

$$\beta = \pi - \phi_m$$

$$\Theta = \pi - 2\beta = 2\phi_m - \pi$$

$$b = R\sin\phi_m = R\sin\frac{\Theta+\pi}{2} = R\cos\frac{\Theta}{2}$$

Then $\quad \frac{d\sigma}{d\Omega} = \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right|$

$$= \frac{1}{\sin\Theta}\left(R\cos\frac{\Theta}{2}\right)\left(\frac{R}{2}\sin\frac{\Theta}{2}\right) = \frac{R^2}{4}$$

indep Θ

Particle equally likely to scatter in any direction

Suggests idea of <u>total</u> cross section

$$\sigma_T = \int \frac{d\sigma}{d\Omega}\, d\Omega$$

Here $\quad \sigma_T = \frac{R^2}{4}\underbrace{\int d\Omega}_{4\pi} = \pi R^2 \quad$ = cross sectional area of sphere

That's where terminology comes from

Generally, σ_T = actual or effective area of scatterer

$= \infty$ for $\frac{1}{r}$ potential

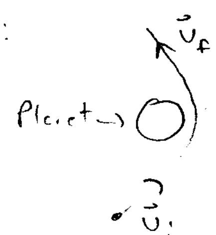
Cross sections not so important in classical physics
Very important in quantum mechanics
→ often only way to probe $V(r)$

Obviously, calculation of $\frac{d\sigma}{d\Omega}$ is different in QM
but concept the same

Finish with another space example
"sling shot" effect
(related to scattering)

= way to boost energy of spaceship
w/o firing rockets

Picture:



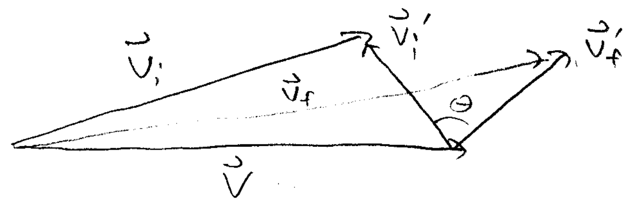
In CM frame, energy is conserved
$v_i = v_f$

In solar frame, planet is moving
velocity $\vec{V}$

Rocket velocity $\vec{v}$ in sun frame
$\vec{v}\,'$ in planet frame

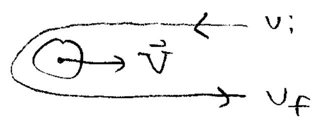
$$\vec{v} = \vec{v}\,' + \vec{V}$$

Have $|\vec{v}_i\,'| = |\vec{v}_f\,'|$, but $|\vec{v}_i| \neq |\vec{v}_f|$



Analysis straightforward, but a bit messy

Easy in 1D:



$$v_i = v_i' + V$$
$$v_f' = -v_i'$$

$$v_f = v_f' + V = -v_i' + V = 2V - v_i$$

If $V \gg v_i$, get a big boost

Energy comes from planet of course

Technique used in all extraterrestrial space missions

Reduces Δv from rockets by about $\times 10$

Wraps up Ch 1

Next time, jump to Ch 3: Lagrangians!

HW due Thursday!