Newton's Laws

1) Inertial frames exist
   - in which particles move with constant $\vec{v}$ unless acted on by force $\vec{F}$

2) In inertial frame $\vec{F} = \frac{d\vec{p}}{dt}$

   $\vec{p} = m\vec{v}$ = momentum

3) If particle 1 exerts force $\vec{F}$ on particle 2, then particle 2 exerts force $-\vec{F}$ on particle 1

For simple problems, apply directly
- assume you know how to do that

Often better to be more sophisticated
Example: conservation laws

Momentum:
In absence of forces, $\vec{p}$ = constant

Angular Momentum:
Chose origin, such that particle has position $\vec{r}$

Define $\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$

Then $$\frac{d\vec{L}}{dt} = m \underbrace{\cancel{\dot{\vec{r}} \times \vec{v}}}_{0} + m\vec{r} \times \dot{\vec{v}}$$

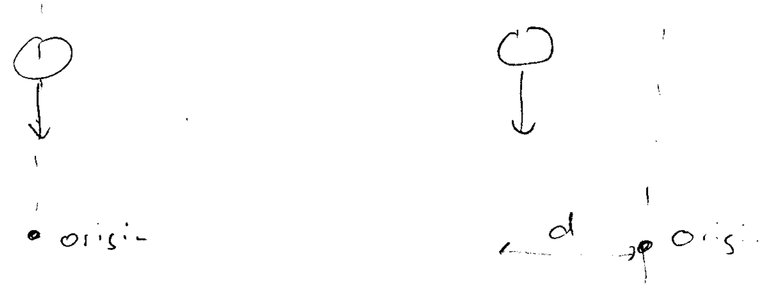
use $m\dot{\vec{v}} = \vec{F}$

$$\boxed{\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} \equiv \vec{\Gamma} = \text{torque}}$$

If $\vec{\Gamma} = 0$, $\vec{L}$ = constant

Note that $\vec{L}$ & $\vec{\Gamma}$ depend on choice of origin

Example: ball falling in gravity



$\vec{\Gamma} = 0$ $\qquad\qquad$ $\vec{\Gamma} = mgd$

$\vec{L}$ = const $\qquad\qquad$ $\vec{L}$ = varying

Energy

Particle moves from $\vec{r}_1$ to $\vec{r}_2$, force $\vec{F}(\vec{r})$

Work $\quad W_{1\to 2} = \int_{r_1}^{r_2} \vec{F}(\vec{r}) \cdot d\vec{s}$

line element of path

$$d\vec{s} = \vec{v}\, dt \, , \qquad \vec{F} = m \frac{d\vec{v}}{dt}$$

$$W_{1\to 2} = \int_{t_1}^{t_2} m \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt = \int_{t_1}^{t_2} \frac{d}{dt}\left(\tfrac{1}{2} m v^2\right) dt$$

$$= \tfrac{1}{2} m v_2^2 - \tfrac{1}{2} m v_1^2$$

Kinetic energy $\quad T = \frac{1}{2} m v^2$

$$\boxed{W_{1\to 2} = T_2 - T_1}$$

If $\quad \vec{F}(\vec{r}) = -\vec{\nabla} U(\vec{r})$ for some U:

$$W_{1\to 2} = -\int_1^2 d\vec{s} \cdot \vec{\nabla} U = -\int_1^2 dU = -U_2 + U_1$$

So $W_{1\to 2} = T_2 - T_1 = U_1 - U_2$

$\Rightarrow \quad T_1 + U_1 = T_2 + U_2$

Total energy $\boxed{E = T + U}$ const

Conservation laws really useful when multiple particles

Use center of mass: $\vec{R} = \frac{1}{M} \sum_i m_i \vec{r}_i$

$M = \sum_i m_i$ = total mass

Force on particle i: $\vec{F}_i = \underbrace{\vec{F}_i^{(e)}}_{\text{external}} + \underbrace{\sum_{j\neq i} \vec{F}_{ji}}_{\text{internal}}$

$M\ddot{\vec{R}} = \sum_i m_i \ddot{\vec{r}}_i = \sum_i \vec{F}_i$

$= \sum_i F_i^{(e)} + \underbrace{\sum_i \sum_{j\neq i} \vec{F}_{ji}}$

each F_{ji} has $F_{ij} = -F_{ji}$, cancels

$$\boxed{M\ddot{\vec{R}} = \vec{F}^{(e)} = \sum_i \vec{F}_i^{(e)}}$$

If don't care about relative motion, acts like one big particle mass M

Define $\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$ total ang momentum

$$\frac{d}{dt}\vec{L} = \sum_i \dot{\vec{r}}_i \times \vec{p}_i + \sum_i r_i \times \dot{p}_i = \sum_i r_i \times \dot{p}_i$$

$$= \sum_i \vec{r}_i \times \left[F_i^{(e)} + \sum_j F_{ji} \right]$$

Consider $\sum_{ij} r_i \times F_{ji} = \frac{1}{2} \sum_{ij} \left(r_i \times F_{ji} + r_j \times F_{ij} \right)$

$$= \frac{1}{2} \sum_{ij} (r_i - r_j) \times F_{ji} \quad \text{(3rd law)}$$

If force between i + j is directed along line between them, then $\Sigma = 0$

Usually true. When not, have a field that itself carries ang. momentum, would need to include

If true, have $\dot{\vec{L}} = \sum_i \vec{r}_i \times \vec{F}_i^{(e)} \equiv \vec{\Gamma}^{(e)}$

Particularly simple if origin located at $\vec{R}$.

$$\vec{r}_i = \vec{R} + \vec{r}_i{}'$$
$$\vec{v}_i = \vec{V} + \vec{v}_i{}'$$

$$\vec{L} = \sum_i (\vec{R} + \vec{r}_i{}') \times m_i (\vec{V} + \vec{v}_i{}')$$

$$= M \vec{R} \times \vec{V} + \underbrace{\left(\sum_i m_i \vec{r}_i{}' \right)}_{0} \times \vec{V} + \vec{R} \times \underbrace{\left(\sum_i m_i \vec{v}_i{}' \right)}_{0} + \sum_i m_i r_i' \times v_i'$$

So $\vec{L} = \vec{R} \times \vec{P} + \sum_i \vec{r}_i{}' \times \vec{p}_i{}'$

$$\equiv \vec{L}_{cm} + \vec{L}'$$

Decompose ang momentum as external (ie, orbital) and internal (ie, spin)

Note $\vec{L}_{cm}$ depends on choice of origin
$\vec{L}'$ always relative to center of mass

Consider torque law

$$\dot{L}_{cm} = R \times \dot{P} = R \times F^{(e)}$$

just like single particle

$$\dot{L}' = \dot{L} - \dot{L}_{cm} = \sum_i (r_i - R) \times F_i^{(e)}$$

$$= \sum_i r_i' \times F_i^{(e)}$$

= torque relative to center of mass

So $\vec{L}'$ = const if no torque about CM (ie, gravity)

Note, this holds regardless of how COM moves

Finally, energy of system

$$\text{Total } T = \sum_i T_i = \sum_i \tfrac{1}{2} m v_i^2$$

$$\vec{v}_i = \vec{V} + v_i'$$

$$v_i^2 = V^2 + v_i'^2 + 2\vec{V}\cdot\vec{v}_i'$$

$$T = \tfrac{1}{2} M V^2 + \sum_i \tfrac{1}{2} m v_i'^2 + \vec{V} \cdot \underbrace{\sum_i m_i v_i'}_{=0}$$

$$T = T_{cm} + T'$$

Conservative force $F_i^{(e)} = -\nabla_i U(r_i)$

$$F_{ji} = -\nabla_i V(r_i - r_j) = -\nabla_{ij} V(r_{ij})$$

for $r_{ij} = r_i - r_j$

Then $W_{1\to 2} = \sum_i \int_1^2 d\vec{s}_i \cdot \vec{F}_i$

$$= \underbrace{\sum_i \int ds_i \cdot F_i^{(e)}}_{\sum_i U(r_i^{(2)}) - U(r_i^{(1)})} + \sum_{ij} \int ds_i \cdot F_{ji}$$

on 2nd term use dummy variable trick

$$\sum_{ij} \int ds_i \cdot F_{ji} = \frac{1}{2} \sum_{ij} \int (d\vec{s}_i - d\vec{s}_j) \cdot \vec{F}_{ji}$$

Note $ds_i - ds_j = dr_{ij}$

So get $-\frac{1}{2} \sum_{ij} dr_{ij} \cdot \nabla_{ij} V(r_{ij})$

$$= -\frac{1}{2} \sum_{ij} \left[V(r_{ij}^{(2)}) - V(r_{ij}^{(1)}) \right]$$

All together,

$$E = T + \sum_i U(\vec{r}_i) + \frac{1}{2} \sum_{i \neq j} V(\vec{r}_{ij}) = \text{const}$$

Again, can separate center of mass and internal motions