Assignment 9

5.3 A tilted coin (a sharp-edged uniform disk) of radius a and mass M rolls without slipping on a horizontal plane in a circle of radius b. A set of orthogonal coordinate axes has its origin at the center of mass, with $\hat{\mathbf{e}}_3$ perpendicular to the face of the coin, $\hat{\mathbf{e}}_2$ in the plane of the coin passing through the point of contact, and $\hat{\mathbf{e}}_1$ parallel to the horizontal plane and tangent to the trajectory. Introduce the angles θ, ϕ, and γ that specify the orientation of the coin as indicated in the figure. (See the text, page 169.) The particular motion of interest (neglecting rolling friction that eventually slows down the coin) is characterized by

$$\dot{\gamma} = \text{const} \qquad \dot{\phi} = \text{const} \qquad \dot{\theta} = 0$$

(hence $\theta = $ const.)

(a) Use the equations of rigid-body motion to eliminate the reaction force at the point of contact to obtain

$$\frac{d}{dt}\mathbf{L}_{\text{cm}} = Ma\left[g\sin\theta - \dot{\phi}^2\cos\theta(b - a\sin\theta)\right]\hat{\mathbf{e}}_1$$

(b) Show, in general, that

$$\boldsymbol{\Omega} = \dot{\theta}\hat{\mathbf{e}}_1 + \dot{\phi}\cos\theta\hat{\mathbf{e}}_2 - \dot{\phi}\sin\theta\hat{\mathbf{e}}_3$$

$$\mathbf{L}_{\text{cm}} = I_1\dot{\theta}\hat{\mathbf{e}}_1 + I_1\dot{\phi}\cos\theta\hat{\mathbf{e}}_2 + I_3(\dot{\gamma} - \dot{\phi}\sin\theta)\hat{\mathbf{e}}_3$$

where $\boldsymbol{\Omega}$ is the instantaneous angular velocity of the (1, 2, 3) body-associated frame as seen in the $1^0, 2^0, 3^0$ center-of-mass frame.

(c) Use the relation between $(d\mathbf{L}/dt)_{cm}$ and $(d\mathbf{L}/dt)_b$, where the subscript b now refers to the (1, 2, 3) frame, to show that the period τ for motion around the circle and the angle of inclination must satisfy the equation

$$\left(\frac{\tau}{2\pi}\right)^2 = \dot{\phi}^{-2} = \frac{\cos\theta}{4g\sin\theta}(6b - 5a\sin\theta).$$

Note: For a flat disk, $I_1 = I_2 = \frac{1}{4}Ma^2$, $I_3 = \frac{1}{2}Ma^2$.

5.9 A symmetric top with one fixed point in a gravitational field moves with its axis nearly vertical ($\beta \ll 1$) and $p_\alpha = p_\gamma$.

(a) Expand the effective potential in Eq. (31.7) through terms of order β^4.

(b) If $p_\gamma^2 > 4I_1Mg\ell$, show that V_{eff} has a minimum at $\beta = 0$. Sketch V_{eff} for small β. Prove that the frequency of small oscillations about this configuration is given by $\Omega^2 = (4I_1^2)^{-1}(p_\gamma - 4I_1Mg\ell)$. Why does this differ from Eq. (31.16) for $\beta_0 = 0$?

(c) If p_γ^2 is slightly smaller than $4I_1Mg\ell$, show that V_{eff} has a maximum at $\beta = 0$ and a minimum at an angle $\beta_0 = (2 - p_\gamma^2/2I_1Mg\ell)^{1/2}$. Sketch V_{eff} for small β. Prove that the frequency of small oscillations about this configuration is given by $\Omega^2 = (2I_1^2)^{-1}(4I_1Mg\ell - p_\gamma)^2$.

5.10 A symmetric top with one fixed point in a gravitational field starts with the initial conditions

$$\dot{\alpha} = 2\left(\frac{Mg\ell}{3I_1}\right)^{1/2} \qquad \beta = 60^\circ$$

$$\dot{\gamma} = (3I_1 - I_3)\left(\frac{Mg\ell}{3I_1I_3^2}\right)^{1/2} \qquad \dot{\beta} = 0$$

(a) Find the conserved momenta p_α and p_γ and the effective potential. Use a sketch of $V_{\rm eff}$ to discuss the qualitative form of the solution $\beta(t)$.

(b) Show that the equation for β can be written

$$\dot{u}^2 = \frac{Mg\ell}{I_1}(1-u)^2(2u-1)$$

where $u = \cos\beta$. Hence derive the explicit solution

$$\sec\beta = 1 + \operatorname{sech}\left[\left(\frac{Mg\ell}{I_1}\right)^{1/2} t\right]$$

and verify the preceding qualitative analysis in part (a). What is the corresponding behavior of $\dot{\alpha}$ and $\dot{\gamma}$?

Packet #8 (a) State Euler's equations of motion, defining all terms precisely.

(b) Define Euler's angles for rigid body motion and express the body components of angular velocity in terms of them.

(c) A symmetrical top, (the moments of inertia are $I_1 = I_2 \neq I_3$) of mass M spins with one point fixed in the earth's gravitational field. Its center of mass is a distance b from the fixed point. Express Euler's equations for the top in terms of Euler's angles and impose the solution of a uniformly precessing top without nutation, i.e. the angle between the figure axis and the vertical direction remains constant. Substituting this solution into the equations of motion, obtain a condition between I_1, I_3, M, b, ϕ, ψ and θ for this solution to be valid.