Assignment 6

2.1 Larmor's Theorem

(a) The Lorentz force implies the equation of motion $m\ddot{\mathbf{r}} = e(\mathbf{E} + c^{-1}\mathbf{v} \times \mathbf{B})$. Prove that the effect of a weak uniform magnetic field $\mathbf{B}$ on the motion of a charged particle in a central electric field $\mathbf{E} = E(|\mathbf{r}|)\hat{\mathbf{r}}$ can be removed by transforming to a coordinate system rotating with an angular frequency $\omega_L = -(e/2mc)\mathbf{B}$. State precisely what "weak" means.

(b) Extend this result to a system of particles of given ratio e/m interacting through potentials $V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)$.

2.2 Assume that over the time interval of interest, the center of mass of the earth moves with approximately constant velocity with respect to the fixed stars and that ω, the angular velocity of the earth, is a constant. Rederive the terrestrial equations of particle motion (11.8) and (11.6) by writing Newton's law in a body-fixed frame with origin at the *surface* of the earth (Fig. 11.2).

2.5 A cannon is placed on the surface of the earth at colatitude (polar angle) θ and pointed due east.

(a) If the cannon barrel makes an angle α with the horizontal, show that the lateral deflection of a projectile when it strikes the earth is $(4V_0^3/g^2)\omega\cos\theta\sin^2\alpha\cos\alpha$, where V_0 is the initial speed of the projectile and ω is the earth's angular-rotation speed. What is the direction of the deflection?

(b) If R is the range of the projectile for the case $\omega = 0$, show that the change in the range is given by $(2R^3/g)^{1/2}\omega\sin\theta\left[(\cot\alpha)^{1/2} - \frac{1}{3}(\tan\alpha)^{3/2}\right]$. Neglect terms of order ω^2 throughout.

2.7 (a) Recall that the centrifugal force $-m\omega \times (\omega \times \mathbf{r})$ at colatitude (polar angle) θ can be written $-m\nabla\Phi_c$, where

$$\Phi_c = -\frac{1}{2}|\omega \times \mathbf{r}|^2 = -\frac{1}{2}\omega^2 r^2 \sin^2\theta$$

If Φ_g is the earth's exterior gravitational potential, prove that a thin surface layer of water (the "ocean") would assume the shape of an equipotential surface $\Phi \equiv \Phi_g + \Phi_c = \text{const}$ (known as the geoid).

(b) Suppose that the earth has a small quadrupole deformation, with

$$\Phi_g = -\frac{M_e G}{r}\left[1 - J_2\left(\frac{R_e}{r}\right)^2 P_2(\cos(\theta)\right]$$

Here $P_2(\cos\theta) = \frac{1}{2}[3\cos^2\theta - 1]$ and $J_2 \approx 1.083 \times 10^{-3}$ characterizes the earth's deviation from sphericity. Take R_e as the radius of the geoid at the equator. Show that the radius of the geoid at colatitude θ is

$$R(\theta) = R_e\left[1 - \cos^2\theta\left(\frac{3}{2}J_2 + \frac{1}{2}\frac{\omega^2 R_e^3}{MG}\right)\right],$$

to lowest order in J_2 and ω. If ΔR is the difference between the geoid's equatorial and polar radii, compare the prediction above with the measured values $(R_e - R_p)/R_e \approx 3.35 \times 10^{-3}$.

(c) Explain why $-\nabla\Phi$ on the geoid is the local acceleration $\mathbf{g}$ sea level, and show that its magnitude is given approximately by $g \approx \partial\Phi/\partial r$. Hence derive Clairaut's formula

$$g(\theta) = g_e\left[1 + \cos^2\theta\left(\frac{5}{2}\frac{\omega^2 R_e^3}{M_e G} - \frac{\Delta R}{R_e}\right)\right]$$

Using the measured value of ΔR, check this result against the measured values $g_e \approx 978.03$ cm/s^2 and $g_p \approx 983.20$ cm/s^2.

Packet #6 If a particle is projected vertically upward from a point on the earth's surface at northern latitude λ, show that it strikes the ground at a point $4/3\omega\cos\lambda\sqrt{8h^3/g}$ to the west (neglect air resistance and consider only small vertical heights).