Assignment 4

4.3 A double pendulum with equal lengths and different masses m_1 and m_2 performs small oscillations in a plane. Introduce the transverse displacements of the first particle from the vertical η_1 and of the second particle from the first particle η_2 (compare Fig. 13.1c).

(a) Show that the lagrangian is given by

$$L = \frac{1}{2}m_1\dot{\eta}_1^2 + \frac{1}{2}m_2\left(\dot{\eta}_1 + \dot{\eta}_2\right)^2 - \frac{g}{2\ell}\left[(m_1+m_2)\,\eta_1^2 + m_2\eta_2^2\right]$$

(b) Derive the normal-mode frequencies $\omega^2 = (g/\ell)(1\pm\gamma)^{-1}$, where $\gamma = [m_2(m_1+m_2)^{-1}]^{1/2}$.

(c) Construct the normal-mode eigenvectors and describe the motions. Show that these represent the expected behavior for large and small values of m_1/m_2.

(d) Verify that the modal matrix has the form

$$\underline{\mathcal{A}} = (2m_1)^{-1/2}\begin{bmatrix} (1-\gamma)^{1/2} & -(1+\gamma)^{1/2} \\ \gamma^{-1}(1-\gamma)^{1/2} & \gamma^{-1}(1+\gamma)^{1/2} \end{bmatrix}$$

and demonstrate explicitly that $\underline{\mathcal{A}}$ diagonalizes the matrices $\underline{m}$ and $\underline{v}$.

(e) Construct the normal coordinates.

(f) Assume that $m_2 \ll m_1$. If the upper mass is displaced slightly from the vertical and released from rest, show that the subsequent motion is such that at regular intervals one pendulum is stationary and the other oscillates with maximum amplitude.

4.4 A particle with mass m slides without friction around the circumference of a circular wire hoop of radius a. The hoop is placed upright in a uniform gravitational field and rotates about a vertical diameter with angular velocity Ω (compare Prob. 3.1).

(a) Construct the lagrangian, using as generalized coordinate the angular displacement θ along the hoop measured from the downward vertical. Derive theh differential equation for the motion and construct the corresponding first integral.

(b) Using the equation of motion, obtain *all* positions of dynamical equilibrium and classify them as stable or unstable. ~~Verify these conclusions by considering forces in the co-rotating frame.~~ For those configurations which are stable, determine the frequency of small oscillations about that position. Discuss the limiting cases $\Omega^2 \ll g/a$ and $\Omega^2 \gg g/a$.

(c) Find the hamiltonian for the system. Is it a constant of the motion? Is it the total energy? Compare with the first integral in part (a).

4.9 (a) A molecule consists of three identical atoms located at the vertices of a 45° right triangle, with equal spring constants k between each pair of atoms. Derive the eigenvalue equation for planar motion and show that it has three degenerate modes at $\omega^2 = 0$. What is their physical interpretation? Find the three remaining nonzero eigenfrequencies.

You need not do part (b).

Packet 4 A particle of mass m attached to one end of a massless spring with spring constant k and relaxed length ℓ. The other end of the spring is attached to a fixed support, and the assembly hangs in a uniform gravitational field of acceleration g.

(a) Write down the Lagrangian and the equations of motion for the particle in three dimensions.

(b) Find the length of the spring and the angle from vertical that are required in order for the particle to execute uniform circular motion about the vertical axis at frequency Ω.

(c) Determine the range of frequencies Ω that can be physically supported, and qualitatively describe the motion of the system as Ω approaches the extremes of this range.

(d) If the particle is displaced slightly from the equilibrium of (b), it will undergo small oscillations. How many distinct frequencies will normally be observed in this motion? No calculation is required here, but do explain the reasoning for your answer.