Assignment 3

3.5 A massless inextensible string passes over a pulley which is a fixed distance above the floor. A bunch of bananas of mass m is attached to one end A of the string. A monkey of mass M is initially at the other end B. The monkey climbs the string, and his displacement $d(t)$ with respect to the end B is a *given* function of time. The system is initially at rest, so that the initial conditions are $d(0) = \dot{d}(0) = 0$. Introduce suitable generalized coordinates and calculate the Lagrangian of the system in terms of these coordinates. Show that the equation of motion governing the height Z of the monkey above the floor is

$$(m + M)\ddot{Z} - m\ddot{d} = (m - M)g$$

Integrate the equation to find the subsequent motions. In the special case that $m = M$, show that the bananas and the monkey rise through equal distances so that the vertical separation between them is constant.

3.14 A particle of mass m suspended by a massless string of length ℓ is stationary in a gravitational field $\mathbf{g}$. It is struck an impulsive horizontal blow, giving it an initial angular velocity ω.

(a) Introduce a Lagrange multiplier and prove the following statements:

1. If $\ell\omega^2 < 2g$, the tension τ does not vanish and the particle does not reach the horizontal.
2. If $2g < \ell\omega^2 < 5g$, the particle passes the horizontal and the string becomes slack before the particle comes to rest.
3. If $5g < \ell\omega^2$, the string always remains taut and the particle executes periodic circular motion.

(b) Discuss the role of the tension in the string by showing how these results are changed if the string is replaced by a rigid massless rod.

3.17 (a) A point mass is placed on the top of a fixed smooth sphere in a gravitational field. Show that it leaves the fixed sphere at a polar angle $\arccos(2/3) \approx 48.19°$.

(b) The point mass is replaced by a roughened sphere that rolls without sliding on the fixed sphere. Show that it leaves the fixed sphere at a polar angle $\arccos(10/17) \approx 53.97°$. Compare these two values with that for a rolling cylinder [see Eq. (19.54)].

3.19 A point mass m is constrained to move without friction on an arbitrary, fixed two-dimensional surface in the absence of external forces. The surface is described by a set of generalized coordinates (q^1, q^2) such that the square of the distance $d\mathbf{s} \cdot d\mathbf{s}$ between two infinitesimally close points on the surface is given by

$$d\mathbf{s} \cdot d\mathbf{s} = ds^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}(q^1, q^2)dq^i dq^j$$

where the *metric tensor* $g_{ij}(q^1, q^2) = g_{ji}(q^1, q^2)$ is symmetric and depends on position.

(a) Construct the lagrangian. Show that the equations of motion for the particle are given by

$$\sum_j g_{ij}\frac{d^2q^j}{dt^2} + \frac{1}{2}\sum_{j,k}\left(\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i}\right)\frac{dq^j}{dt}\frac{dq^k}{dt} = 0 \qquad i = 1, 2$$

Introduce the inverse $(g^{-1})_{ij} \equiv g^{ij}$ of the metric tensor, and derive the equivalent equations of motion

$$\frac{d^2q^i}{dt^2} + \sum_{j,k}\Gamma^i_{jk}\frac{dq^j}{dt}\frac{dq^k}{dt} = 0 \qquad i = 1, 2$$

where the *affine connection* is defined by

$$\Gamma^i_{jk} \equiv \frac{1}{2}\sum_m g^{im}\left(\frac{\partial g_{mj}}{\partial q^k} + \frac{\partial g_{mk}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^m}\right)$$

(b) Show that the curves of minimum distance between two points on the surface, the *geodesics*, are given by

$$\frac{d^2q^i}{d\tau^2} + \sum_{j,k}\Gamma^i_{jk}\frac{dq^j}{d\tau}\frac{dq^k}{d\tau} = 0 \qquad i = 1,2$$

where $0 \leq \tau \leq 1$ is a uniform parametrization of the distance along the curve.

(c) Show that $\mathbf{v}^2 \equiv v^2$ =const for any motion on this surface. For a given trajectory of the particle passing through two points on the surface a distance ℓ apart, prove that we can take $\tau = (v/\ell)t = \text{const} \times t$. Hence conclude that the equations and curves in a and b are identical. *Note:* These observations form one of the starting points for the theory of general relativity.

Packet #3 Under the influence of gravity, a bead of mass m slides without friction down a wire that has the form of a simple curve in a vertical plane – say the xz-plane. The bead starts at the point $(x_i, z_i) = (a, h)$ and ends at the point $(x_f, z_f) = (b, 0)$. Find the curve joining these points (the brachistochrone) for which the bead reaches the end point in the least time, starting from rest at the initial point.