Assignment 12

7.5 A string of uniform mass density and length ℓ hangs under its own weight in the earth's gravitational field. Consider small transverse displacements $u(x,t)$ in a plane.

(a) Compute the equilibrium tension in the string $\tau(x)$, where x is the distance from the point of suspension.

(b) Show that the normal modes satisfy Bessel's equation. *Hint:* See Appendix C and make the substitution $s^2 \equiv \ell - x$.

(c) What are the boundary conditions?

(d) What are the normal-mode frequencies? (Find the first three from tables.)

(e) Express the normal modes $\rho_n(x)$ explicitly as functions of x and the Bessel-function zeros z_n.

7.9 A string of length $2a$ is stretched to a constant tension τ with its ends fixed. The density of the string is given by $\sigma(x) = \sigma_0(1 - |x|/a)$ for $|x| < a$.

(a) Use a zero-parameter trial function to derive a variational estimate of the lowest resonant frequency ω_1. Compare with the numerical value of $\omega_1^2 \approx 3.477\tau/a^2\sigma_0$. (See hints.)

(b) Devise a one-parameter trial function and show that it leads to a better (lower) estimate.

(c) Repeat part (a) for the next eigenfrequency ω_2, whose numerical value is $\omega_2^2 \approx 18.956\tau/a^2\sigma_0$.

7.10 In some cases, the closed form (43.29) of G_ω can be constructed only for $\omega = 0$. This problem illustrates that the resulting G_0 is still useful.

(a) Consider the Sturm-Liouville equation $L_0\rho = \omega^2\sigma\rho$, where L_0 is given in Eq. (43.4a) and ρ satisfies appropriate boundary conditions at the ends of the interval $x = a$ and $x = b$. Define a Green's function $G_0(x, x')$ by the equation [compare Eq. (43.12) for $\omega = 0$]

$$L_0 G_0(x, x') = \delta(x, x')$$

plus appropriate boundary conditions. Show that ρ satisfies a *homogeneous* integral equation

$$\rho(x) = \omega^2 \int_a^b G_0(x, x')\sigma(x')\rho(x')dx'$$

(b) Verify that the functional

$$\omega^2[\rho] = \frac{\int dx\, \sigma(x)\rho^2(x)}{\iint dx\, dx'\, \rho(x)\sigma(x)G_0(x, x')\sigma(x')\rho(x')}$$

is stationary for small variations in ρ about the exact solution.

(c) Apply this variational principle to the system in Prob. 7.9 and obtain an upper bound for the lowest frequency using your zero-parameter trial function. How does the estimate compare with those obtained in Prob. 7.9?

S5 Make up and solve a problem that you think would be suitable for the final exam. It can be on any topic we have covered. The problem should not be particularly difficult (and you won't get any more credit for doing a hard problem than an easy one), but don't make it completely trivial. As a guide, it should be hard enough that you couldn't expect someone solve it in their head without writing anything down.