Assignment 11

6.16 Consider a particle of mass m moving in three dimensions in the attractive potential $V(r) = -k/r$ with k =const. Use spherical polar coordinates (r, θ, ϕ).

(a) Prove that the energy can be expressed in terms of the action variables as

$$E = -\frac{2\pi^2 mk^2}{(J_r + J_\theta + J_\phi)^2}$$

(b) Discuss the frequencies of the system.

(c) Use the Sommerfeld-Wilson quantization conditions to derive the Bohr theory of the hydrogen atom.

6.17 Consider a canonical transformation generated by the function

$$S(q_1, ..., q_n, P_1, ..., P_n, t) = \sum_\sigma q_\sigma P_\sigma + \epsilon G(q_1, ..., q_n, P_1, ..., P_n, t)$$

where ϵ is an infinitesimal quantity.

(a) Show that the resulting canonical tranform differs from the identity by terms of order ϵ with

$$P_\sigma = p_\sigma - \epsilon \left(\frac{\partial G}{\partial q_\sigma} \right)_p + O(\epsilon^2)$$

$$Q_\sigma = q_\sigma + \epsilon \left(\frac{\partial G}{\partial p_\sigma} \right)_q + O(\epsilon^2)$$

(b) Under this canonical transformation, show that an arbitrary function $F(q_1, ..., q_n, p_1, ..., p_n)$ changes by an amount $dF = \epsilon[F, G]_{PB}$.

(c) If G is a constant of the motion with $\partial G/\partial t = 0$, prove that the hamiltonian H is invariant under such transformations. Discuss the necessary symmetry of H if G is a component of the the total linear or angular momentum along some axis.

6.18 Use the fundamental Poisson-bracket relations (37.9) and (37.10) to derive the following Poisson brackets involving the components of the angular momentum

$$[L_i, L_j]_{PB} = L_k \qquad i, j, k \text{ in cyclic order}$$
$$[L^2, L_i]_{PB} = 0$$

In the treatment of central forces, we took L_z as the canonical momentum p_ϕ for the variable ϕ. Can the set L_x, L_y, L_z serve as canonical momenta for some set of generalized coordinates?

S4 Prove the following properties of Poisson brackets, for arbitrary functions F_1, F_2 and G:

(a) $[F_1, F_1]_{PB} = 0$

(b) $[F_1 + F_2, G]_{PB} = [F_1, G]_{PB} + [F_2, G]_{PB}$

(c) $[F_1 F_2, G]_{PB} = F_1[F_2, G] + F_2[F_1, G]_{PB}$

In particular, (c) implies that $[F^2, G]_{PB} = 2F[F, G]_{PB}$.

Packet #10 A particle of mass m and energy E moving in one dimension comes in from $-\infty$ and encounters the repulsive potential:

$$U(x) = \frac{U_0}{\cosh^2 \alpha x}$$

where U_0 and α are given parameters, and the particle energy $E > U_0 > 0$.

Compute the delay time for this potential. In other words, how much longer will it take for the particle to travel from $x = \infty$ to $x = +\infty$ as compared to the travel time for free motion at the same energy? (The travel time for finite L may be defined as the time it takes to travel from L to $+L$. Find an expression for this with and without the potential, and then find the limit of the time difference as $L \to \infty$.)

State what happens to the delay time as the particle energy E approaches (i) U_0 and (ii) infinity, and explain the physics behind these two types of behavior.

You might need the following integral:

$$\int \frac{1}{\sqrt{x^2 + bx + x}} dx = \ln \left| 2x + b + 2\sqrt{x^2 + bx + c} \right| + \text{const}$$