Assignment 10

6.4 The relativistic motion of a particle in a static potential $V(\mathbf{r})$ can be obtained from the lagrangian $L = -mc^2(1 - v^2/c^2)^{1/2} - V(\mathbf{r})$.

(a) Write out Lagrange's equations and verify the above assertion.

(b) Find the canonical momentum $\mathbf{p}$ and show that the relativistic Hamiltonian becomes $H = (m^2c^4 + p^2c^2)^{1/2} + V(\mathbf{r})$. Is H a constant of the motion?

(c) Assume V is spherically symmetric. Show that $\mathbf{r} \times \mathbf{p}$ is a constant of the motion. Hence reduce H to the form $H = c(m^2c^2 + p_r^2 + r^{-2}p_\phi^2)^{1/2} + V(r)$.

6.5 A particle with charge e and mass m moves in a specified electromagnetic field with $\Phi = 0$ and $\mathbf{A} = \hat{\mathbf{z}}A_z(x, y, t)$.

(a) Use the Lorentz-force equation (33.8) to construct the explicit first integral $\dot{z} + eA_z/mc = C$ (a constant).

(b) Show that the x and y equations can then be written $\ddot{\mathbf{r}}_\perp = -\nabla_\perp \frac{1}{2}(C - eA_z/mc)^2$.

(c) Consider a uniform static magnetic field $\mathbf{B} = B_0\hat{\mathbf{x}}$. Integrate these equations to show that the particle executes a helix about the $\hat{\mathbf{x}}$ direction with an angular frequency eB_0/mc. (To be specific, assume initial conditions $\mathbf{r}(0) = 0$, $\dot{\mathbf{r}}(0) = (v_{0x}, v_{0y}, v_{0z})$. Find the radius and center of the helix.)

6.8 For a system described by cartesian coordinates, use the results of Prob. 6.7 to prove that the functions $S_0 + \mathbf{P} \cdot d\mathbf{r}$ and $S_0 + \hat{\mathbf{n}} \cdot \mathbf{L} d\phi$ generate infinitesimal translations $d\mathbf{r}$ and rotations $\hat{\mathbf{n}} d\phi$, respectively, where $\mathbf{P}$ and $\mathbf{L}$ are the total linear and angular momenta.

6.12 A point mass m with initial position $\mathbf{r}_0$ and velocity $\mathbf{v}_0$ moves in a uniform gravitational field. Use the Hamilton-Jacobi method to reduce the motion to find both the time-dependent trajectory and the orbit.

Packet #9 A particle of mass m is constrained to move on the surface of a torus (solid thick ring), shown below, but otherwise moves freely. (There is no gravity in this problem. The torus is floating somewhere in intergalactic space.) Points on the surface can be represented by a pair of angles (ψ, θ) such that

$$x = (a + b\cos\theta)\cos\psi y = (a + b\cos\theta)\sin\psi z = b\sin\theta \tag{1}$$

where $a > b$.

The particle is initially traveling around the outermost equatorial circle ($\theta = 0$) with velocity v. But then it is given a very, very tiny impulse in the θ-direction, so that it starts to oscillate while it continues to transverse the equatorial circle (as shown by the wiggling line on the torus below). Find the frequency of these oscillations in θ, treating θ as small.

[Hint helpful for certain derivations: If you find any equation of motion of the form d[“something”]$/dt = 0$, then note that “something” is a constant of the motion.]