Assignment 6 Hints

4.11 – The hardest part here is getting the potential energy right. You should be able to use the same approach as for problem 4.9, namely

$$V_{j,j+1} = \frac{1}{2}k\left(|\mathbf{r}_{j+1} - \mathbf{r}_j| - a_0\right)^2$$

or

$$V_{j,j+1} = \frac{1}{2}k\left(\sqrt{(X_{j+1} - X_j)^2 + (Y_{j+1} - Y_j)^2} - a_0\right)^2$$

if you use (X_j, Y_j) for the position of particle j. You can expand this using $X_j = aj + x_j$ and $Y_j = y_j$ for small deviations x_j, y_j. But here you'll need to expand the square root to second order to get the right second-order expansion of V.

To show that the transverse and longitudinal modes decouple, show that the equation of motion for x_j depends only on other x's, and the equation of motion for y_j depends only on other y's. (It is generally true that if $L(q_1, q_2) = L(q_1) + L(q_2)$, then q_1 and q_2 decouple. You could just prove that instead if you like.)

4.13 – The dispersion relation (see page 115) is just the functional relation $\omega(k)$ relating the frequency to the wave number. When sketching the dispersion relation, you can take $g/l = k/m$.

4.15 – I'm a little puzzled by this problem. Parts (c) and (d) seem pretty strongly to suggest that you can't get an analytical expression for the allowed values of k, but I was able to do so without much difficulty. I checked my answer against the $N = 2$ and $N = 3$ cases and I think I have it right. So perhaps the authors made a mistake here.

I followed the problem's suggestion and just got the equations of motion for each particle using Newton's law. Then I tried a solution of the form

$$\eta_j = Ae^{i(kx_j - \omega t)} + Be^{i(-kx_j - \omega t)}$$

with k and ω to be determined. For $0 < j < N$, I have the same equation of motion as in (24.4), so from the book or notes you know this requires

$$\omega^2 = \frac{2\kappa}{m}\left(1 - \cos ka\right).$$

The $j = 0$ and $j = N$ equations are different, and give a pair of equations to solve for B/A and k (A itself remains as a normalization constant that you don't need to determine). Note that you'll need to plug the above expression for ω into these equations to get a solution.

Assuming you get a simple answer like I did, all you need to do for part (c) is show that the solutions have the claimed symmetry under $j \leftrightarrow N - j$. You can skip the rest of (c) and (d).

4.17 – Solve this by setting up three equations: a wave equation for the string on $0 < x < \ell/2$, a wave equation for $\ell/2 < x < \ell$, and the equation of motion for the mass. The force on the mass is the y-component of the tension in the strings, so the three equations are coupled. Look for a solution for all three systems with time dependence $\cos(\omega t + \phi)$. The strings will have spatial solutions $u \propto \cos(kx + \phi)$. Use the boundaries at the ends of the strings and at the middle to determine the phases ϕ and the allowed values of k. The equations of motion then relate ω and k.

You don't need to do part (c).