Assignment 11 Hints

6.16 – Note that here we are *not* taking the angular momentum $\mathbf{L}$ to be along the z axis, making this a more general problem than what we've done before.

The first and hardest step is to calculate $J_\sigma = \oint p_\sigma dq_\sigma$ for each σ. Write out H and the Hamilton-Jacobi equation. Try a solution $S = W_r(r) + W_\theta(\theta) + W_\phi(\phi) - Et$, and use the separability of the system obtain three equations

$$\frac{dW_\phi}{d\phi} = \alpha_\phi$$
$$\left(\frac{dW_\theta}{d\theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = \alpha_\theta^2$$
$$\frac{1}{2m}\left[\left(\frac{dW_r}{dr}\right)^2 + \frac{\alpha_\theta^2}{r^2}\right] - \frac{k}{r} = E$$

Use $p_\sigma = dW_\sigma/dq_\sigma$ and solve these for the relations $p_\sigma(q_\sigma)$. The first is trivial, and the third involves an integral you can look up. The second integral is hard and it's not in my tables. I'll walk you through it, but you should fill in the steps.

You should get

$$J_\theta = \oint \sqrt{\alpha_\theta^2 - \frac{\alpha_\phi^2}{\sin^2\theta}}d\theta$$

The integral is over one cycle of the motion. In one half cycle, θ varies from θ_{min} to θ_{max}, with $\theta_{\text{min}} = \sin^{-1}(\alpha_\phi/\alpha_\theta)$ and $\theta_{\text{max}} = \pi - \theta_{\text{min}}$. So the integral can be expressed as

$$2\alpha_\theta \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} \left(1 - \frac{\sin^2\theta_{\text{min}}}{\sin^2\theta}\right)^{1/2} d\theta.$$

Substituting $\zeta = \theta - \pi/2$, this can be rewritten as

$$4\alpha_\theta \int_0^{\zeta_0} \left(1 - \frac{\cos^2\zeta_0}{\cos^2\zeta}\right)^{1/2} d\zeta = 4\alpha_\theta \int_0^{\zeta_0} \frac{1}{\cos\zeta}\left(\sin^2\zeta_0 - \sin^2\zeta\right)^{1/2}$$

for $\zeta_0 = \cos^{-1}(\alpha_\phi/\alpha_\theta)$. Then substitute $\sin\zeta = \sin\zeta_0 \sin\psi$ to get

$$4\alpha_\theta \sin^2\zeta_0 \int_0^{\pi/2} \frac{\cos^2\psi}{1 - \sin^2\zeta_0 \sin^2\psi}\psi.$$

Finally, take $u = \tan\psi$ to get

$$4\alpha_\theta \sin^2\zeta_0 \int_0^\infty \frac{du}{(1+u^2)(1+u^2\cos^2\zeta_0)},$$

which can be integrated using the method of partial fractions, or you can look it up.

Once you have the J's as functions of the α's and E, invert to get E and you should have the result for (a). The rest of the problem is pretty straightforward. For part (c), refer to pages 198-199 in the text. The book doesn't discuss it, but you might check whether you get the right degeneracies for the energy levels, as well as the correct energies.

6.17 – This is a generalization of what you did in Problem 6.8 and problem 6.7 that I did in class on 11/6. It should not be too hard, just make sure to keep the p's and P's straight.

As far as I can tell, the problem in the book has an error, since you need to specify $\partial G/\partial t = 0$ in part (c).

6.18 – Do this using Cartesian coordinates and $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. If you use the results of problem S3, then the only hard part is keeping track of the indices... if you want, just do $[L_x, L_y]_{PB}$ and generalize from there using the cyclic symmetry of the cross product.

S3 – These are pretty trivial, and shouldn't give you any trouble. I mostly just thought they would be useful for 6.18.