Phys 521 Final Exam 10 December 2008

This is a closed book, closed notes exam, to be taken in a single three-hour period. The problems should be worked on separate pages and attached to this sheet when completed. There are six problems, which will be weighted equally. For full credit, be sure to show and explain all your work.

Name: ______________________________

Signature: ______________________________

Some possibly useful formulas:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\int_0^1 \frac{dx}{\sqrt{1 - x^4}} = 1.311$$

$$\int_0^1 \sqrt{1 - x^4}\, dx = 0.874$$

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right)$$

1. A bead of mass m slides without friction on a wire in the shape of a parabola, described by

$$y = -a + \frac{x^2}{2a}.$$

In addition, a spring with spring constant k stretches from the mass to a fixed point at the origin $x = y = 0$. For simplicity, take the unstretched length of the spring to be zero.

(a) Choose a suitable generalized coordinate and calculate the Lagrangian.

(b) Determine the equation of motion.

(c) Find all equilibrium points and calculate the frequency for small oscillations about each.

2. For the system of problem 1,

(a) Use conservation of energy to obtain an integral form for the solution of the motion.

(b) For values of the energy slightly above the minimum possible value, use the integral from (a) to determine the period of the motion explicitly, as a function of the energy, m, k, and a. Check that your answer is consistent with what you obtained for problem 1(c).

3. The normal modes of a stretched string satisfy the Sturm-Liouville equation

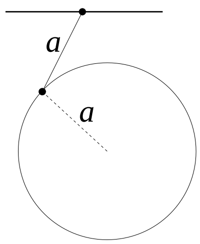
$$-\frac{d}{dx}\left(\tau \frac{d\rho}{dx}\right) + v\rho = \omega^2 \sigma \rho$$

for functions $\tau(x)$, $v(x)$, and $\sigma(x)$. Here x ranges from a to b. For boundary conditions $d\rho/dx|_a = \alpha\rho(a)$ and $d\rho/dx|_b = \beta\rho(b)$, prove that two modes ρ_n and ρ_m are orthogonal according to

$$\int_a^b \rho_n(x)\rho_m(x)\sigma(x)dx = 0$$

as long as $\omega_n^2 \neq \omega_m^2$.

4. A pendulum is constructed from a massless string of length a attached to a sphere of mass m and radius a, as shown. Calculate the two frequencies exhibited for small displacements from equilibirium. Recall that the moment of inertia for a sphere about its center is $(2/5)ma^2$.



5. Parabolic coordiates (ξ, η, ϕ) are defined in terms of cylindrical coordinates (ρ, z, ϕ) by

$$z = \frac{1}{2}(\xi - \eta) \qquad \text{and} \qquad \rho = \sqrt{\xi\eta}.$$

The angular coordinate ϕ is the same in both systems.

(a) Calculate the Lagrangian using parabolic coordinates, for a particle of mass m moving in the central potential $V = \gamma/r = \gamma/\sqrt{\rho^2 + z^2}$.

(b) Calculate the Hamiltonian $H(p, q)$, expressed in terms of the canonical momenta (p_ξ, p_η, p_ϕ).

(c) Show that the Hamilton-Jacobi equation

$$H\left(\left\{\frac{\partial S}{\partial q_\sigma}\right\}, \{q_\sigma\}, t\right) + \frac{\partial S}{\partial t} = 0$$

is separable in this case, and write out the three resulting equations with their separation constants. (You do **not** need to solve these equations.)

6. (a) A mechanical system with Lagrangian L is in a uniform gravitational field $\mathbf{g}$. An additional external force causes the system as a whole to accelerate horizontally (perpendicular to $\mathbf{g}$) with constant acceleration a. Show that in a coordinate frame accelerating with the system, there is an apparent gravitational field $\mathbf{g}'$ at an angle θ to $\mathbf{g}$. Determine $|\mathbf{g}'|$ and θ.

(b) Use the above result to find the oscillation frequency of a simple pendulum of length ℓ that is attached to a support that is moving horizontally with constant acceleration a.