

Supplement 4 - Partial Wave Expansion of e^{ikz}

In spherical coordinates, $e^{ikz} = e^{ikr \cos \theta}$

Let $x = kr$

Now we can write $e^{ix \cos \theta} = \sum_l c_l j_l(x) P_l(\cos \theta)$

Since j_l is only solution to radial Sch. Eqn. that is regular at origin

Use orthogonality of P_l 's: $\int_0^\pi P_l(\cos \theta) P_{l'}(\cos \theta) \sin \theta d\theta = \frac{2 \delta_{ll'}}{2l+1}$

$$\int P_l(\cos \theta) e^{ix \cos \theta} \sin \theta d\theta = \frac{2}{2l+1} c_l j_l(x)$$

Now we could just set $x=0$, and get

$$c_l = \frac{2l+1}{2} \frac{1}{j_l(0)} \int P_l(\cos \theta) \sin \theta d\theta$$

except that integral and $j_l(0)$ are both zero for $l > 0$

Instead, note $j_l(x) \rightarrow \frac{2^l l!}{(2l+1)!} x^l$ for small x
(Table 4.4)

$$\text{So } \left(\frac{d}{dx}\right)^l j_l(x) = \frac{2^l (l!)^2}{(2l+1)!}$$

$$\text{also have } \left(\frac{d}{dx}\right)^l e^{ix \cos \theta} = i^l \cos^l \theta e^{ix \cos \theta}$$

So, differentiating both sides and then setting $x=0$,

$$\int P_l(\cos \theta) i^l \cos^l \theta \sin \theta d\theta = \frac{2^{l+1} (l!)^2}{(2l)!} \frac{1}{(2l+1)^2} c_l$$

$$c_l = \frac{(2l)!}{2^{l+1} (l!)^2} (2l+1)^2 i^l \int P_l(\cos \theta) \cos^l \theta \sin \theta d\theta$$

Define $y = \cos \theta$

$$\text{Integral becomes } I = \int_{-1}^1 y^l P_l(y) dy$$

Use Rodrigues formula, Eq 4.28

$$P_l(y) = \frac{1}{2^l l!} \left(\frac{d}{dy} \right)^l (y^2 - 1)^l$$

$$I = \frac{1}{2^l l!} \int_{-1}^1 y^l \left(\frac{d}{dy} \right)^l (y^2 - 1)^l dy$$

Integrate by parts: $u = y^l \quad du = l y^{l-1} dy$
 $dv = \left(\frac{d}{dy} \right)^l (y^2 - 1)^l \quad v = \left(\frac{d}{dy} \right)^{l-1} (y^2 - 1)^l$

$$I = \frac{1}{2^l l!} \left[\underbrace{y^l \left(\frac{d}{dy} \right)^{l-1} (y^2 - 1)^l}_{\downarrow} \Big|_{-1}^1 - l \int_{-1}^1 y^{l-1} \left(\frac{d}{dy} \right)^{l-1} (y^2 - 1)^l dy \right]$$

Since power of y^{l-1} is greater than power of $\frac{d}{dy}$, will have y^{l-1} term at end $\rightarrow 0$

$$I = \frac{1}{2^l l!} (-l) \int_{-1}^1 y^{l-1} \left(\frac{d}{dy} \right)^{l-1} (y^2 - 1)^l dy$$

Repeating l times, see that

$$\begin{aligned} I &= \frac{1}{2^l l!} (-1)^l l! \int_{-1}^1 (y^2 - 1)^l dy \\ &= \frac{1}{2^l} \int_{-1}^1 (1 - y^2)^l dy \end{aligned}$$

To do this one, use integration by parts again

$$\text{In general, consider } J_{mn} = \int_{-1}^1 y^m (1 - y^2)^n dy$$

$$\begin{aligned} u &= (1 - y^2)^n & du &= -2yn(1 - y^2)^{n-1} \\ v &= \frac{y^{m+1}}{m+1} & dv &= y^m \end{aligned}$$

$$\begin{aligned} \text{So } J_{mn} &= \underbrace{(1-y^2)^n \frac{y^{m+1}}{m+1} \Big|_{-1}^1}_{=0} + 2 \frac{n}{m+1} \int_{-1}^1 y^{m+2} (1-y^2)^{n-1} dy \\ &= 2 \frac{n}{m+1} J_{(m+2)(n-1)} \end{aligned}$$

$$\begin{aligned} \text{We have } I &= \frac{1}{2^l} J_{0l} \\ &= \frac{1}{2^l} 2 \frac{l}{1} J_{(2)(l-1)} \\ &= \frac{1}{2^l} 2^2 \frac{l(l-1)}{1 \cdot 3} J_{(4)(l-2)} \\ \text{repeating, } &= \frac{1}{2^l} 2^l \frac{l!}{1 \cdot 3 \cdot 5 \cdots (2l-1)} J_{(2l)(0)} \end{aligned}$$

$$\text{Note } 1 \cdot 3 \cdot 5 \cdots (2l-1) = \frac{(2l)!}{2 \cdot 4 \cdot 6 \cdots 2l} = \frac{(2l)!}{2^l l!}$$

$$\begin{aligned} \text{So } I &= \frac{1}{2^l} 2^l \frac{2^l (l!)^2}{(2l)!} \int_{-1}^1 y^{2l} dy \\ &= \frac{2^l (l!)^2}{(2l)!} \frac{2}{2l+1} \end{aligned}$$

Plugging back in, get

$$\begin{aligned} c_l &= \frac{(2l)!}{2^{l+1} (l!)^2} (2l+1)^2 i^l \cdot \frac{2^{l+1} (l!)^2}{(2l)!} \frac{1}{2l+1} \\ &= (2l+1) i^l \end{aligned}$$

$$\text{Thus } e^{ikz} = \sum_l i^l (2l+1) j_l(kr) P_l(\cos \theta) \quad \text{as claimed}$$