Supplement 2: Proof that # of states is conserved when combining angular momenta

Suppose we're combining J_1 and J_2, with $J_2 > J_1$

In uncoupled basis have $(2J_1+1)(2J_2+1)$ states

In coupled basis, have $J = J_2 - J_1,\ J_2 - J_1 + 1,\ \ldots\ J_2 + J_1$

so total number of states is

$$Q = \sum_{J=J_2-J_1}^{J_2+J_1} (2J+1)$$

Do this sum: $Q = 2 \sum_{J_2-J_1}^{J_2+J_1} J + \sum_{J_2-J_1}^{J_2+J_1}$

in second sum, have $2J_1+1$ terms, so sum $= 2J_1+1$

for first sum, use $\sum_{n=0}^{N} n = \frac{N(N+1)}{2}$

So $\sum_{J_2-J_1}^{J_2+J_1} J = \sum_{0}^{J_2+J_1} J - \sum_{0}^{J_2-J_1-1} J$

$$= \frac{(J_2+J_1)(J_2+J_1+1)}{2} - \frac{(J_2-J_1-1)(J_2-J_1)}{2}$$

$$= \frac{1}{2}\Big[J_2^2 + J_1^2 + 2J_1J_2 + J_1 + J_2 - \left(J_2^2 + J_1^2 - 2J_1J_2 - J_2 + J_1\right)\Big]$$

$$= \frac{1}{2}\left[4J_1J_2 + 2J_2\right]$$

$$= 2J_1J_2 + J_2$$

So we get $Q = 4J_1J_2 + 2J_2 + 2J_1 + 1$

$$= (2J_1+1)(2J_2+1)$$

same as uncoupled basis.