Lecture 8

Last time, introduced basic idea of statistical mechanics

1) Figure out what configurations are possible, given macroscopic constraints
2) Count how many states included in each configuration
3) Determine probabilities by assuming all states equally likely

Today, do this in general! Suppose N particles, energy E

in potential w/ single particle energy levels = $E_1, E_2, E_3, \ldots$

Also, allow each level to be degenerate with $d_1, d_2, d_3, \ldots$ single particle states

For example, spin $\frac{1}{2}$ electrons in cubical box
ground state $d_1 = 2$ (spin)
1st excited state $d_2 = 6$ (excitation in $x, y,$ or z) $\times$ (spin)

Label configurations with occupation numbers
$[N_1, N_2, N_3, \ldots]$

So N_n = # of particles in level n.

Define $Q(N_1, N_2, \ldots)$ = # of states corresponding to given configuration.

Need to calculate Q, for given exchange symmetry.

Distinguishable particles

Start with $n = 1$, N_1 particles

How many ways to pick N_1 particles from total N?

N choices for first
$N-1$ for 2^{nd}
$N-2$ for 3^{rd}
$\vdots$
$N-N_1+1$ for N_1^{th}

So $N(N-1)(N-2)\cdots(N-N_1+1) = \dfrac{N!}{(N-N_1)!}$ choices total

But it doesn't matter what order I pick those N_1 particles in.

ie, particle B could come first or last, No difference

We counted those possibilities separately, so we over counted. Need to divide out number of different permutations of the N_1 selected.

How many?
N_1 choices for 1^{st}
N_1-1 2^{nd}
$\vdots$
1 for last

So $N_1(N_1-1)\ldots 1 = N_1!$ permutations.

So really have $\frac{N!}{N_1!(N-N_1)!}$ distinct ways to select N_1 particles out of N

$$\equiv \binom{N}{N_1}$$, binomial coefficient (often say as "N choose N_1")

Now, each of these N_1 particles can go in any of d_1 different states

So $d_1^{N_1}$ more choices.

$\Rightarrow$ total possibilities = $\frac{N! \, d_1^{N_1}}{N_1!(N-N_1)!}$ for $n=1$ level

For $n=2$ level, everything is the same, except only $N-N_1$ particles left to choose from,

So have $\frac{(N-N_1)! \, d_2^{N_2}}{N_2!(N-N_1-N_2)!}$ possibilities

similar for higher n's.

To get total Q, multiply possibilities for each n

$$Q = \frac{N! \, d_1^{N_1}}{N_1!(N-N_1)!} \times \frac{(N-N_1)! \, d_2^{N_2}}{N_2!(N-N_1-N_2)!} \times \frac{(N-N_1-N_2)! \, d_3^{N_3}}{N_3! \cdots} \times \cdots$$

$$= N! \frac{d_1^{N_1} d_2^{N_2} \cdots}{N_1! N_2! \cdots} = \boxed{N! \prod_{n=1}^{\infty} \frac{d_n^{N_n}}{N_n!}}$$

OK, now do for bosons:

Don't need to worry about picking N_1 particles from N.
Just one way, since particles are identical

Quantum state will be superposition of all $\binom{N}{N_1}$ choices.

But harder to decide how many ways to distribute N_1 particles over d_1 states

Book shows one way, I'll show another (Problem 5.25):

For given d consider various N_1s:

$N_1 = 1$: d ways

$N_1 = 2$: Could put both in same state: d ways
<u>or</u> in diff states : $d(d-1)/2$ ways

Total $= d + \frac{1}{2}d(d-1) = \frac{1}{2}d(d+1)$ ways

$N_1 = 3$: All in same state: d ways
Two in one, one in another: $d(d-1)$ ways
Three separate: $d(d-1)(d-2)/3!$

$$\text{Total} = d + d(d-1) + \frac{1}{6}d(d-1)(d-2) = \frac{1}{6}d(d+1)(d+2)$$

When faced with a progression like this, guess that

generally have $\dfrac{(d+N_1-1)!}{(d_n-1)!\,N_1!}$ ways $= \binom{N_1+d-1}{N_1}$

Mathematician would prove by induction. Physicist finds clever argument like in book.

For Q, just multiply factors for each n:

$$\boxed{Q_{boson} = \prod_{n=1}^{\infty} \frac{(N_n+d_n-1)!}{N_n!\,(d_n-1)!}}$$

Finally, fermions pretty easy.
Only one way to pick N_1 particles

Distribute over d_1 states, one particle per state ...

d_1 choices for first
d_1-1 . for 2^{nd}
$\vdots$
d_1-N_1+1 for N_1^{th}

$$= \frac{d_1!}{(d_1-N_1)!}$$

Divide by $N_1!$ to account for permutations

$$\rightarrow \frac{d_1!}{(d_1-N_1)!\,N_1!} = \binom{d_1}{N_1}$$

So $$\boxed{Q_{fermions} = \prod_{n} \frac{d_n!}{N_n!\,(d_n-N_n)!}}$$

Note if $N_n > d_n$, configuration not possible

OK: interpret $k! = \infty$ for $k<0$

So we did it... we have $Q(N_1, N_2, \ldots)$

What do we do with this?

Basic plan: assume that N is really big, $\sim 10^{23}$

Then Q's are totally humongous.

Also turns out that Q is pretty sharply peaked around its maximum.

If $[N_1^*, N_2^*, \ldots]$ is config with largest $Q = Q^*$ then significantly different configs have $Q \ll Q^*$

To get idea of why:
flip a coin N times. Most likely get $\frac{N}{2} \pm \sqrt{N}$ heads

For $N \to \infty$, fraction of heads $\to \frac{1}{2} \pm \frac{1}{\sqrt{N}} \to \frac{1}{2}$
Becomes very likely to get close to 50% heads.
= most likely configuration

Generalize: say physical system most likely to be in config $[N_1^*, N_2^* \ldots]$

So we want to find N_n^*

But two constraints: Need $\sum_n N_n = N$
and $\sum_n N_n E_n = E$

See how to handle that next time.