Lecture 8

Last time, introduced basic idea of statistical mechanics

1) Figure out what configurations are possible, given microscopie constraints
2) Count how many states included in each configuration
3) Determine probabilities by assuming all states equally likely

Today, do this in general! Suppose N particles, energy E

in potential well single particle energy levels: \( E_1, E_2, E_3, \ldots \)

Also, allow each level to be degenerate
with \( d_1, d_2, d_3, \ldots \) single particle states

\[
\begin{align*}
\text{For example, spin-}\frac{1}{2} \text{ electrons in cubic box} \\
\text{ground state } d_1 = 2 \quad \text{(spin)} \\
\text{1st excited state } d_2 = 6 \quad \text{(excitation in } x, y, z) \times 1/\text{spin)}
\end{align*}
\]

Label configurations with occupation numbers
\([N_1, N_2, N_3, \ldots]\)

\( N_n \) = # of particles in level \( n \).

Define \( Q(N_1, N_2, \ldots) \) = # of states corresponding to given configuration.

Need to calculate \( Q \), for given exchange symmetry.
Distinguishable particles

Start with \( n = 1 \), \( N \), particles

How many ways to pick \( N_i \) particles from total \( N \)?

- \( N \) choices for first
- \( N-1 \) for 2nd
- \( N-2 \) for 3rd
- \( \ldots \)
- \( \ldots \)
- \( N-N_i+1 \) for \( N_i \)

So \( \frac{N!}{(N-1)! (N-2)! \ldots (N-N_i+1)!} \) choices total

But it doesn't matter what order I pick those \( N \), particles in.

So, particle B could come first or last, No difference.

We counted those possibilities separately, so we over counted. Need to divide out number of different permutations of the \( N_i \) selected.

How many?

- \( N_i \) choices for 1st
- \( N_i-1 \) for 2nd
- \( \ldots \)
- \( \ldots \)
- \( 1 \) for last

So \( N_i (N_i-1) \ldots 1 = N_i! \) permutations.
So really have \( \frac{N!}{N_1!(N-N_1)!} \) distinct ways to select \( N_1 \) particles out of \( N \)

\[ \equiv \binom{N}{N_1}, \text{ binomial coefficient} \]

(often say as "N choose \( N_1 \)"

Now, each of these \( N_1 \) particles can go in any of \( d_1 \) different states

So \( d_1^{N_1} \) more choices,

\[ \Rightarrow \text{ total possibilities} = \frac{N! \cdot d_1^{N_1}}{N_1!(N-N_1)!} \quad \text{for } n=1 \text{ level} \]

For \( n=2 \) level, everything is the same, except only \( N-N_1 \) particles left to choose from,

So have \( \frac{(N-N_1)! \cdot d_2^{N_2}}{N_2!(N-N_1-N_2)!} \) possibilities

similar for higher \( n \)'s.

To get total \( Q \), multiply possibilities for each \( n \)

\[ Q = \frac{N! \cdot d_1^{N_1}}{N_1!(N-N_1)!} \times \frac{(N-N_1)! \cdot d_2^{N_2}}{N_2!(N-N_1-N_2)!} \times \frac{(N-N_1-N_2)! \cdot d_3^{N_3}}{N_3!} \times \ldots \]

\[ = \frac{d_1^{N_1} \cdot d_2^{N_2} \ldots}{N_1! \cdot N_2! \ldots} = \prod_{n=1}^{N} \frac{d_n^{N_n}}{N_n!} \]
OK, now do for bosons:

Don't need to worry about picking $N$ particles from $N$. Just one way, since particles are identical.

Quantum state will be superposition of all $\binom{N}{N_i}$ choices.

But harder to decide how many ways to distribute $N_i$ particles over $d_i$ states.

Book shows one way, I'll show another (Problem 5.25).

For given $d$ consider various $N_i$'s:

$N_i = 1$: $d$ ways

$N_i = 2$: Could put both in same state: $d$ ways

Or in diff states: $d(\frac{d}{2})$ ways

Total = $d + \frac{d}{2}d(\frac{d}{2}) = \frac{d}{2}d(d+1) = \frac{d}{2}d(d+1)$ ways

$N_i = 3$: All in same state: $d$ ways

Two in one, one in other: $d(\frac{d}{3})$ ways

Three separate: $d(\frac{d}{6})d(d-1)/3!$

Total = $d + \frac{d}{3}d(d-1) + \frac{1}{6}d(d-1)(d-2) = \frac{1}{6}d(d+1)(d+2)$

When faced with a progression like this, guess that

Generally have $\binom{d+N_i-1}{d-1}$ ways: $\binom{N_i(d-1)}{N_i}$

Mathematician would prove by induction. Physicist finds clever argument like in book.
For Q, just multiply factors for each r:

\[
Q_{\text{config}} = \prod_{n=1}^{m} \frac{(N_n+d_n-1)!}{N_n! (d_n-1)!}
\]

Finally, fermions pretty easy.
Only one way to pick \( N_i \) particles

Distribute over \( d_i \) states, one particle per state

\[
d_i! \text{ choices for first }
\]
\[
d_{i-1}! \text{ for } \ldots
\]
\[
\ldots d_1! \text{ for } N_1^{(1)}
\]
\[
= \frac{d_i!}{(d_i-N_i)!}
\]

Divide by \( N_i! \) to account for permutations

\[
\frac{d_i!}{(d_i-N_i)! N_i!} = \binom{d_i}{N_i}
\]

So

\[
Q_{\text{config}} = \prod_n \frac{d_n!}{N_n! (d_n-N_n)!}
\]

Note if \( N_n > d_n \), configuration not possible

OK: interpret \( k! = \infty \) for \( k < 0 \)

So we did it... we have \( Q/N_1, N_2, \ldots \)
What do we do with this?

Basic plan: assume that $N$ is really big, $N \approx 10^{23}$

Then $Q^*$'s are totally homogeneous.

Also turns out that $Q^*$ is pretty sharply pecked around its maximum.

If $\left[ N_1^*, N_2^*, \ldots \right]$ is config with lowest $Q = Q^*$

then significantly different configs have $Q \geq Q^*$

To get idea of why:

flip a coin $N$ times. Most likely $\frac{N}{2} = \sqrt{N}$ heads

For $N_1^*$, fraction of heads $= \frac{1}{2} \pm \sqrt{N}^{1/2}

Becomes very likely to get close to 50% heads = most likely configuration

Generalize: say physical system most likely to be in configs $\left[ N_1^*, N_2^*, \ldots \right]$.

So we want to find $N^*_a$

But two constraints: $N_a \geq 0$ and $\sum N_a = N$

and $\sum N_a E_a = E$

See how to handle that next time.