Lecture 4

Last time introduced many-body wave functions

\[ \Psi(r_1, r_2, t) \]

For identical particles, introduce exchange operator \( \pi \):

\[ \pi \Psi(r_1, r_2) = \Psi(r_2, r_1) \]

Claim that any two identical particles must be in eigenstate of \( \pi \)

even: \[ \Psi_+ (r_1, r_2) = A \left[ 2 \phi_a(r_1) \phi_b(r_2) + 2 \phi_b(r_1) \phi_a(r_2) \right] \]

= integral spins, = bosons

odd: \[ \Psi_-(r_1, r_2) = A \left[ 2 \phi_a(r_1) \phi_b(r_2) - 2 \phi_b(r_1) \phi_a(r_2) \right] \]

= half-integer spins, = fermions

Means that identical quantum particles are truly indistinguishable... fundamentally no way to tell one electron from another
(because the one you're looking at might actually be the other!)

Let's do a problem: 5.4

What is normalization constant \( A \) in \( \Psi_+ ? \)

a) Assuming \( \phi_a \& \phi_b \) are normalized and orthogonal
For fun, let's use Direct notation

\[ |\Psi_+\rangle = A \left( |\Psi_a\rangle_1, |\Psi_b\rangle_2 \pm |\Psi_b\rangle_1, |\Psi_a\rangle_2 \right) \]

We can \[ \langle \Psi_a | \Psi_a \rangle = 1 \]

\[ = |A|^2 \left[ \left( \langle \Psi_a | \Psi_a \rangle + \langle \Psi_b | \Psi_b \rangle \right) (|\Psi_a\rangle_1, |\Psi_b\rangle_2 \pm |\Psi_b\rangle_1, |\Psi_a\rangle_2) \right] \]

\[ = |A|^2 \left[ \langle \Psi_a | \Psi_a \rangle (|\Psi_a\rangle_1, |\Psi_b\rangle_2 + |\Psi_b\rangle_1, |\Psi_a\rangle_2) \right.

\[ \left. \pm \langle \Psi_b | \Psi_b \rangle (|\Psi_a\rangle_1, |\Psi_b\rangle_2 + |\Psi_b\rangle_1, |\Psi_a\rangle_2) \right) \]

We're assuming \[ \langle \Psi_a | \Psi_b \rangle = \langle \Psi_b | \Psi_a \rangle = 1 \]

\[ \langle \Psi_a | \Psi_b \rangle = 0 \]

So \[ 1 = |A|^2 \left[ 1 + 1 \right] = 2 |A|^2 \Rightarrow |A| = \sqrt{\frac{1}{2}} \]

In wavefunction notation, \[ |\Psi_+\rangle (r_1, r_2) = \sqrt{\frac{1}{2}} \left[ |\Psi_a(r_1)\Psi_b(r_2)\rangle \pm |\Psi_b(r_1)\Psi_a(r_2)\rangle \right] \]

b) What if \[ |\Psi_a\rangle = |\Psi_b\rangle \]

For bosons, have \[ |\Psi_+\rangle = A \left( |\Psi_a\rangle, |\Psi_a\rangle + |\Psi_a\rangle, |\Psi_a\rangle \right) \]

\[ = 2A |\Psi_a\rangle, |\Psi_a\rangle \rangle_2 \]

So \[ 1 = \langle \Psi_+ | \Psi_+ \rangle = 4|A|^2 \langle \Psi_a | \Psi_a \rangle \rangle_1 \langle \Psi_a | \Psi_a \rangle \rangle_2 \]

\[ = 4|A|^2 \Rightarrow \left[ A = \frac{1}{2} \right] \]

or \[ |\Psi_+\rangle = |\Psi_a\rangle_1, |\Psi_a\rangle_2, |\Psi_a\rangle \]

\[ \Psi \equiv |\Psi_a(r_1)\Psi_a(r_2)\rangle \]

\[ \Psi \equiv |\Psi_a(r_1)\Psi_a(r_2)\rangle \]
For fermions, \[ |2 \text{fermions} \rangle = A \left( |2 \text{e}_1 \rangle |2 \text{e}_2 \rangle - |2 \text{e}_2 \rangle |2 \text{e}_1 \rangle \right) = 0 \]

Can't normalize zero

No physical wave function,

Conclude that we can't have two fermions in the same quantum state!

This is the Pauli exclusion principle

In fact, because of exchange there is an effective "repelling force" between two fermions, and attractive between two bosons.

Sometimes called the "exchange interaction"

Can get idea by considering two particles in a box, in states \( \psi_a(x) \) and \( \psi_b(x) \)

(assume \( \psi_a \) \& \( \psi_b \) orthonormal = normalized)

Work out mean square distance between particles

\[ \Delta^2 = \langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2 \langle x_1 x_2 \rangle \]

Discussion (on pg. 3)
First, suppose particles are distinct (electron spin, for example).

Take \( \Psi(x_1, x_2) = \Psi_a(x_1) \Psi_b(x_2) \)

\( 12\rangle = \langle 12_a \rangle_1 \langle 22_b \rangle_2 = \langle a\rangle_1 \langle b\rangle_2 \)

Then \( \langle x_1^2 \rangle = \langle 24 \mid x_1^2 \mid 12\rangle \)

\[ = \langle 12 \mid x_1^2 \mid 16 \rangle \langle 16 \mid 1a \rangle \]

\[ = \langle a \mid x_1^2 \mid a \rangle \underline{\langle b \mid b \rangle}_2 \]

\[ = \langle a \mid x_1 \mid a \rangle \underline{\langle b \mid b \rangle}_2 \]

\[ = \langle a \mid x_1 \mid a \rangle = \int dx_1 x_1^2 |\Psi_a(x)|^2 \]

\[ = \langle x^2 \rangle_a \quad \text{mean square position for particle } a \]

Similarly, get \( \langle x_2^2 \rangle = \langle x^2 \rangle_b \)

Then \( \langle x_1 x_2 \rangle = \langle 12 \mid 51 \mid x_1, x_2 \mid 1a \rangle \langle 1a \rangle \underline{\langle 1b \rangle_2} \)

\[ = \langle a \mid x_1 \mid a \rangle \langle b \mid x_2 \mid b \rangle \]

\[ = \langle x \rangle_a \langle x \rangle_b \]

Put together, \( \Delta^2 = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2 \langle x \rangle_a \langle x \rangle_b \)

\[ = \Delta_a^2, \quad \text{distinguishable} \]
Compared to identical particles:
\[
|12\rangle = \frac{1}{\sqrt{2}} \left( |1a\rangle, 1b\rangle + 1b, 1a\rangle \right) = \frac{1}{\sqrt{2}} \left( |ab\rangle + 1a\rangle \right)
\]
Still have \(\Delta^2 = \langle x_1^2 \rangle + \langle x_1^2 \rangle - 2\langle x_1 x_2 \rangle\)

Now \(\langle x_1^2 \rangle = \frac{1}{2} \left( \langle a, b | x_1 | a, b \rangle \langle a, b | x_1 | a, b \rangle \right)\)
\[= \frac{1}{2} \left( \langle a, b | x_1 | a, b \rangle \langle a, b | x_1 | a, b \rangle \right) + \langle b, a | x_1 | b, a \rangle \langle b, a | x_1 | b, a \rangle\]
\[= \left( \langle x_1^2 \rangle \right)_a + \left( \langle x_1^2 \rangle \right)_b\]

By symmetry, \(\langle x_1^2 \rangle = \langle x_2^2 \rangle\)

But \(\langle x_1 x_2 \rangle = \frac{1}{2} \left( \langle a, b | x_1 x_2 | a, b \rangle \langle a, b | x_1 x_2 | a, b \rangle \right)\)
\[= \frac{1}{2} \left( \langle a, b | x_1 x_2 | a, b \rangle \langle a, b | x_1 x_2 | a, b \rangle \right) + \langle b, a | x_1 x_2 | b, a \rangle \langle b, a | x_1 x_2 | b, a \rangle\]
\[= \left( \langle x_1 x_2 \rangle \right)_a + \left( \langle x_1 x_2 \rangle \right)_b\]
\[\pm \left( \langle x_1 x_2 \rangle \right)_{ab} \langle x_1 x_2 \rangle \right]_{ab} \pm \left( \langle x_1 x_2 \rangle \right)_{ab} \langle x_1 x_2 \rangle \right]

For \(\langle x \rangle_{ab} = \langle a | x | b \rangle = \int \phi_a^*(x) \phi_b(x) \, dx\)

So \(\langle x_1 x_2 \rangle = \langle x \rangle_a \langle x \rangle_b \pm \left| \langle x \rangle_{ab} \right|^2\)
\[ \Delta^2 = \langle x^2 \rangle + \langle y^2 \rangle - 2 \langle x \rangle \langle y \rangle \]
\[ = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2 \langle x \rangle_a \langle x \rangle_b + 2 |\langle x \rangle_a \rangle^2 \]
\[ \text{or} \quad \Delta^2 = \Lambda_d^2 + 2 |\langle x \rangle_a \rangle^2 \]

\[ \Lambda_d = \text{separation for distinguishable particles,} \]

Since \( |\langle x \rangle_a \rangle^2 \geq 0 \), have \( \Delta^2 < \Lambda_d^2 \) for bosons

\[ \Delta^2 > \Lambda_d^2 \] for fermions

**Bosons pushed together**

**Fermions pushed apart**

\[ \text{Related to normalization issue:} \]

**Two fermions located at same } x \text{ have same amplitude to be found in the same position eigenstate} \ldots \text{ but this is not allowed!} \]

\[ \text{Note that } \quad 21_i (x, x) = \frac{1}{\sqrt{2}} \left[ 2 \frac{1}{e}(x) \right] \left( \frac{1}{e}(x) - \psi \frac{1}{e}(x) \right) \]
\[ = 0 \]

**Fermions can't get close to each other.**

But notice that if \( \langle x \rangle_a b = 0 \), then \( \Delta = \Lambda_d \)

no effects from exchange.

\( \text{When is } \langle x \rangle_a b = 0 \) ?

\( \text{When } \psi_a \text{ a } \psi_b \text{ are never nonzero in the same place} \)
\( = \text{) when they don't overlap} \)

**Generally true: if wave functions don't overlap, don't worry about exchange symmetry.**