Lecture 36

Integral eqn vs of Schr eqn $\quad \psi(\vec{r}) = \frac{2m}{\hbar^2} \int G(|\vec{r}-\vec{r}_0|)\, V(\vec{r}_0)\, \psi(\vec{r}_0)\, d^3 r_0$

$$G(r) = -\frac{1}{4\pi r} e^{ikr}$$

Note, $\psi(\vec{r}) = 0$ if $V = 0$

not right. Really can add any solution to free particle equation $(\nabla^2 + k^2)\psi_0 = 0$

$$\psi(\vec{r}) = \psi_0(\vec{r}) + \frac{2m}{\hbar^2} \int G V \psi \, d^3 r_0$$

Ready to apply to scattering

Interested in limit $r \gg$ range of V

But integral limited to region $r_0 \leq$ range of V

So we can take $|\vec{r}_0| \ll |\vec{r}|$ in

$$G(|\vec{r}_0 - \vec{r}_0|) = -\frac{1}{4\pi} \frac{1}{|\vec{r}-\vec{r}_0|} e^{ik|\vec{r}-\vec{r}_0|}$$

Taylor expand:

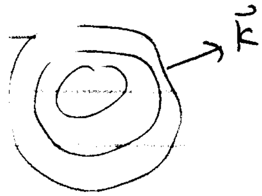
$$|\vec{r}-\vec{r}_0|^2 = r^2 + r_0^2 - 2\vec{r}\cdot\vec{r}_0$$

$$= r^2\left(1 - \frac{2\vec{r}\cdot\vec{r}_0}{r^2} + \frac{r_0^2}{r^2}\right)$$

$$\simeq r^2\left(1 - \frac{2\vec{r}\cdot\vec{r}_0}{r^2}\right)$$

So $|\vec{r}-\vec{r}_0| \approx r\left(1 - \frac{\vec{r}\cdot\vec{r}_0}{r^2}\right) = r - \hat{r}\cdot\vec{r}_0$

Lets define $\vec{k} = k\hat{r}$ = local wave vector of scattered wave



Then $e^{ik|\vec{r}-\vec{r}_0|} \approx e^{ikr} e^{-i\vec{k}\cdot\vec{r}_0}$

Note that even for $r \to \infty$, could have $kr_0 \gg 1$

Can't neglect term in exp

Outside exponential, can afford to be less accurate:

$$\frac{1}{|\vec{r}-\vec{r}_0|} = \frac{1}{r - \hat{r}\cdot\vec{r}_0} = \frac{1}{r}\Big(\frac{1}{1-\underbrace{\frac{\hat{r}\cdot\vec{r}_0}{r}}_{\to 0 \text{ as } r\to\infty}}\Big) \to \frac{1}{r}$$

Then $G(|\vec{r}-\vec{r}_0|) \to -\frac{1}{4\pi}\frac{e^{ikr}}{r} e^{-i\vec{k}\cdot\vec{r}_0}$

and

$$\psi \to \psi_0 - \frac{m}{2\pi\hbar^2}\underbrace{\frac{e^{ikr}}{r}}_{\text{scattered wave}} \int e^{-i\vec{k}\cdot\vec{r}_0} V(\vec{r}_0)\,\psi(\vec{r}_0)\,d^3r_0$$

Take $\psi_0 = Ae^{ikz}$ to be incident wave

Born approximation: if V is small, then $\psi \approx \psi_0$

Take $\psi \to \psi_0$ in integral

$$\psi \to A\Big[e^{ikz} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\vec{k}\cdot\vec{r}_0} V(\vec{r}_0)\, e^{ikz_0}\, d^3r_0\Big]$$

Define $\vec{\varkappa} = k\hat{z} - \vec{k} = k(\hat{z} - \hat{r})$

$= \vec{k}_{incident} - \vec{k}_{scattered}$

Then $\psi \to A\left\{e^{ikz} - \left[\frac{m}{2\pi\hbar^2}\int e^{i\vec{\varkappa}\cdot\vec{r}_0} V(\vec{r}_0)\, d^3r_0\right]\frac{e^{ikr}}{r}\right\}$

Compare to standard form

$$\psi \to A\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right]$$

See $$\boxed{f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{i\vec{\varkappa}\cdot\vec{r}_0} V(\vec{r}_0)\, d^3r_0}$$

This is basic Born approx result
Simplifies in a few special cases

I: $U(\vec{r})$ is spherically symmetric (usual case)

Take $\vec{r}_0$ axes aligned to $\hat{z}$, so $\vec{\varkappa}\cdot\vec{r}_0 = \varkappa r_0 \cos\theta$

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{i\varkappa r_0\cos\theta_0} V(r_0)\, 2\pi r_0^2 \sin\theta_0\, d\theta_0\, dr_0$$

$$= -\frac{m}{\hbar^2}\int r_0^2 V(r_0)\left[\int_0^\pi e^{i\varkappa r_0\cos\theta_0}\sin\theta_0\, d\theta_0\right]$$

$$\left[\frac{2\sin\varkappa r_0}{\varkappa r_0}\right]$$

$$\boxed{f(\theta) = -\frac{2m}{\hbar^2\varkappa}\int_0^\infty r_0 V(r_0)\sin\varkappa r_0\, dr_0}$$

Here θ comes in through $\varkappa$:

$$\vec{\varkappa} = k(\hat{z} - \hat{r})$$

$$\varkappa^2 = k^2(1+1-2\hat{z}\cdot\hat{r}) = 2k^2(1-\cos\theta)$$
$$= 4k^2\sin^2\frac{\theta}{2}$$

$$\boxed{\varkappa = 2k\sin\frac{\theta}{2}}$$

This is usually the best form to use

II If $V(r)$ is short range compared to $\frac{1}{k}$

Say $kr_0 \ll 1$ whenever $V(r_0) \neq 0$

Then $\varkappa r_0 \ll 1$, take $e^{i\vec{\varkappa}\cdot\vec{r}_0} \to 1$ in integral

Get $$\boxed{f(\theta) = -\frac{m}{2\pi\hbar^2}\int V(\vec{r}_0)\,d^3r_0}$$

low energy limit:
if range of potential is R,
need $k \ll \frac{1}{R}$

See that $f(\theta)$ independent of θ
$\Rightarrow$ makes sense, expect s-wave scattering as $k \to 0$

Example: Yukawa potential
$$V(r) = \beta\,\frac{e^{-\mu r}}{r}$$

"Screened" Coulomb interaction
Important in plasma physics, and
nuclear/high energy

Spherically symmetric, so use

$$f(\theta) = -\frac{2m\beta}{\hbar^2 \varkappa}\left\{\int_0^\infty e^{-\mu r}\sin\varkappa r\,dr\right\}$$

$$\left\{\frac{1}{2i}\int_0^\infty e^{(-\mu+i\varkappa)r} - e^{(-\mu-i\varkappa)r}\,dr\right\}$$

$$\left\{\frac{1}{2i}\left[\frac{e^{(-\mu+i\varkappa)r}}{-\mu+i\varkappa} - \frac{e^{(-\mu-i\varkappa)r}}{-\mu-i\varkappa}\right]\Big|_0^\infty\right\}$$

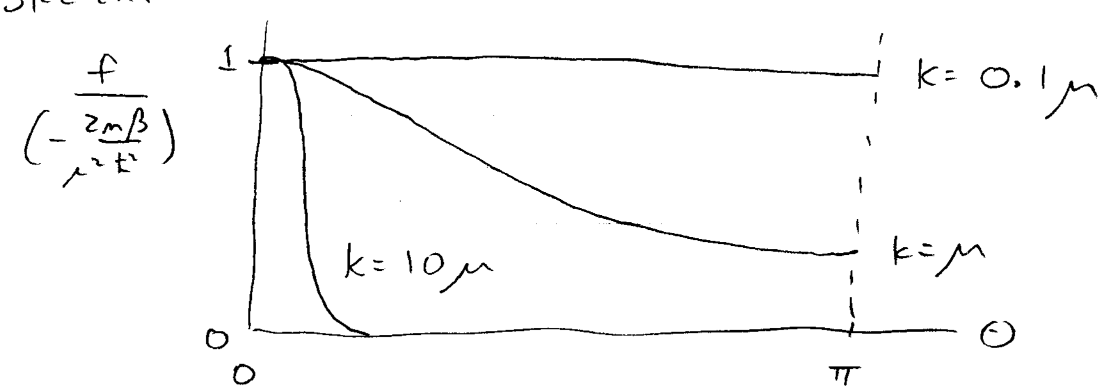
$$\left\{\frac{1}{2i}\left[\frac{-1}{-\mu+i\varkappa} - \frac{-1}{-\mu-i\varkappa}\right]\right\}$$

$$\left\{\frac{1}{2i}\left[\frac{1}{\mu-i\varkappa} - \frac{1}{\mu+i\varkappa}\right]\right\}$$

$$\frac{1}{2i}\left[\frac{2i\varkappa}{\mu^2+\varkappa^2}\right]$$

$$\boxed{f(\theta) = -\frac{2m\beta}{\hbar^2}\frac{1}{\mu^2+\varkappa^2}} \qquad \text{again, } \varkappa = 2k\sin\frac{\theta}{2}$$

Sketch:



As energy increases, backward scattering ($\theta = \pi$) $\to 0$

forward scattering $\to$ constant

In general: don't get much backward scattering if $k \gg \frac{1}{R}$

Remember, when Rutherford saw large angle scatterings, he knew that atoms has small nuclei: R was smaller than expected.

Finish by expressing total cross section

$$\sigma = \int |f(\theta)|^2 d\Omega = 2\pi \int |f|^2 \sin\theta \, d\theta$$

Notice that we always get f as fn of $\varkappa$

So change variable to $\varkappa = 2k \sin\frac{\theta}{2}$

$$d\varkappa = k \cos\frac{\theta}{2} \, d\theta$$

But $\sin\theta = 2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}$

$$\text{so} \quad \sin\theta \, d\theta = \left(2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}\right)\left(\frac{d\varkappa}{k \cos\frac{\theta}{2}}\right)$$

$$= \frac{2}{k} \sin\frac{\theta}{2} \, d\varkappa$$

$$= \frac{\varkappa}{k^2} \, d\varkappa$$

So $\boxed{\sigma = \frac{2\pi}{k^2} \int_0^{2k} |f(\varkappa)|^2 \varkappa \, d\varkappa}$

Here $$\sigma = \frac{2\pi}{k^2} \left(\frac{2m\beta}{\hbar^2}\right)^2 \int_0^{2k} \frac{\varkappa \, d\varkappa}{(\mu^2+\varkappa^2)^2}$$

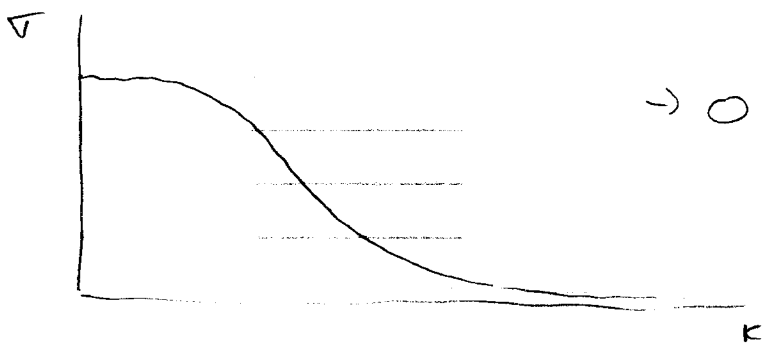
$$= \frac{\pi}{k^2} \left(\qquad \right)^2 \int_0^{4k^2} \frac{du}{(\mu^2+u)^2} \qquad u = \varkappa^2$$

$$= \frac{\pi}{k^2} \left(\qquad \right)^2 \left[-\frac{1}{(\mu^2+u)} \Big|_0^{4k^2} \right]$$

$$\sigma = \frac{\pi}{k^2}\left(\frac{2m\beta}{\hbar^2}\right)^2\left[\frac{1}{\mu^2} - \frac{1}{\mu^2+4k^2}\right]$$

$$\left[\frac{4k^2}{\mu^2(\mu^2+4k^2)}\right]$$

$$\boxed{\sigma = \pi\left(\frac{4m\beta}{\hbar^2\mu}\right)^2\frac{1}{\mu^2+4k^2}}$$



$\to 0$ as $k \to \infty$

make sense,

since $\frac{V}{E} \to 0$