Transitions in atoms  \( H' = -e \phi + -\gamma E \)  

Using TOPT, get

For coherent drive  \( P_{a \rightarrow b} = \left| \frac{\gamma E_0 \hbar}{\hbar} \sin^2 \frac{\Delta t}{2} \right|^2 \)

where  \( \Delta = \omega - \omega_0 \)

For incoherent drive  \( R'_{a \rightarrow b} = \frac{i \hbar}{\omega_0} \left| \gamma E_0 \right|^2 \rho(\omega_0) \)

\( \rho = \) spectral density

What about polarization of light in this case? Most sources of natural light are not polarized, not just \( \rho_z \), that matters.

Really write  \( R_z = \frac{i \hbar}{\omega_0} \left| \rho_z \right|^2 \rho_z(\omega_0) \)

\( \rho_z = \) energy density of \( z \)-polarized light

Similarly  \( R_x = \frac{i \hbar}{\omega_0} \left| \rho_x \right|^2 \rho_x(\omega_0) \)

\( \rho_x = \) ...

But by symmetry of atom, have  \( \left| \rho_x \right|^2 = \left| \rho_y \right|^2 = \left| \rho_z \right|^2 \)

unless we have specifically prepared atom in a non-symmetric state

(a particular m-level, for instance)
Usually we have not done that

In that case \( |\overline{\rho}|^2 = |\rho_x|^2 + |\rho_y|^2 + |\rho_z|^2 = 3|\rho_z|^2 \)

\( \overline{\rho} = < b' \cdot e^{a^2} > \)

So \( R_i = \frac{\pi}{3\varepsilon_0 c^2} |\overline{\rho}|^2 \rho_i(\omega_0) \)

Total rate

\[ R = \sum R_i = \frac{\pi}{3\varepsilon_0 c^2} |\overline{\rho}|^2 \sum \rho_i(\omega_0) \]

\[ R = \frac{\pi}{3\varepsilon_0 c^2} (|\overline{\rho}|^2 \rho)(\omega_0) \]

where now \( \rho = \sum \rho_i \) = total spectral density

Example:

Sunlight has \( \rho = 10^{-25} \, \text{J/s} \, \text{m}^{-2} \)

at \( \lambda = 122 \text{nm} \), at radius of Earth's orbit

Last time got:

\[ \sum \rho_x = \frac{2^{15}}{3^{10}} e^{2\alpha^2} \] for \( 1s \rightarrow 3 \rho \)

So

\[ R_{1s \rightarrow 3 \rho} = \frac{\pi}{3\varepsilon_0 c^2} \times 3 \times \frac{2^{15}}{3^{10}} e^{2\alpha^2} \rho \]

\[ |\overline{\rho}|^2 = 3|\rho_z|^2 \]

Plug in, get \( 10^{-4} \, \text{s}^{-1} \)

So a hydrogen atom in Earth's upper atmosphere is excited about every \( \lambda = 3 \) hours
However, it doesn't stay excited that long.

**Observed fact:**
excited quantum states spontaneously decay
to ground state, even if no driving field

**Why should that be?**

Excited state = eigenstate of $\mathbf{H}$
should be stationary

But we left out part of the Hamiltonian:

EM field is a quantum system itself

We've treated field as classical up to now

So if we say $\mathbf{E} = 0$, then it is

But can't really do that, any more then we
can place particle at $\mathbf{P} = 0$

... always some quantum uncertainty

So when we set $\mathbf{E} = 0$, really still a
"zero-point field" present

This field acts as $\mathbf{H}'$, drives spontaneous emission

---

**Can we calculate rate?**
Yes

But need to develop QM theory for fields
vs. particles: quantum field theory

Not really too hard, but don't really have
time in this class
Happily, we can get rate in a different way that requires no QFT.

Method developed by Einstein, 1916 (!)

Suppose two-level system, in equilibrium with black-body field at temperature T.

So incoherent radiation present, with

\[ \rho(\omega_0) = \rho_0 = \frac{1}{\pi^2 c^3} \frac{\omega_0^3}{e^{\hbar \omega_0 / kT} - 1} \]

from Planck's black body formula.

Then two processes occur:

"stimulated" transitions \( a \rightarrow b \)

driven by thermal radiation

\[ \text{Rate } R = \frac{\pi}{3 \hbar \omega_0^2 / 2} \rho_0 = B \rho_0 \]

"spontaneous" transitions \( b \rightarrow a \)

Unknown rate \( A \)

Suppose \( N_a \) atoms in state \( a \)

\( -N_b \) in \( b \)

Then we know

\[ \frac{dN_b}{dt} = -AN_b - B \rho_0 N_b + B \rho_0 N_a \]

spont.

stim.

stim.

emission.

emission.

absorption.
Follows from definition of transition rate - called "rate equation"

In equilibrium know \( \frac{dN_b}{dt} = 0 \)

So \( (A + B\rho_o)N_b = B\rho_o N_a \)

\[
\frac{N_b}{N_a} = \frac{B\rho_o}{A + B\rho_o}
\]

But from stat mech (assume distinguishable particles)

have \( \frac{N_b}{N_a} = e^{-(E_b - E_a)/k_B T} = e^{-\frac{\hbar \omega_b}{k_B T}} \)

Put in for \( \rho_o \):

\[
e^{-\frac{\hbar \omega_b}{k_B T}} = \frac{B}{A + B} \frac{\frac{\hbar^2}{\pi^2 c^3}}{e^{\frac{\hbar \omega_b}{k_B T}} - 1}
\]

Solve for \( A \), get \( A = \frac{\omega_o^3 \frac{\hbar^2}{\pi^2 c^3} B}{\hbar^2} \)

or

\[
A = \frac{\omega_o^3 \frac{\hbar^2}{3 \pi \varepsilon_0 c^3}}{3 \pi \varepsilon_0 c^3}
\]

Pretty clever!

Example, \( A_{\gamma-p} \approx A \approx \frac{\omega_o^3}{3 \pi \varepsilon_0 c^3} \left( \frac{2}{3} \frac{2 \pi^2}{310} \varepsilon^2 a^2 \right) \)

\[
= 6.2 \times 10^{-8} \text{ s}^{-1}
\]

Pretty fast... lifetime in excited state \( \tau = \frac{1}{A} = 166 \text{ ns} \)
Say more next time.

Now tie back to idea of zero-point field
problem 9.9 John Lillegren

Note, we can solve for excited state population:

\[ \frac{dN_b}{dt} = -A N_b \]

If \( p_1 = 0 \) (no driving field), rate eqs are

Solution \( N_b(t) = N_b(0) e^{-At} \)

So \( 1/A = \text{exponential decay constant} \)
\( \rightarrow \) "lifetime"

\[ \]