Lecture 25

Talk about actual TDPT

Derived basic eqns for 2-level system:

\[ \dot{C}_a = -\frac{i}{\hbar} H_{ab} e^{-i\omega t} C_b \]
\[ \dot{C}_b = -\frac{i}{\hbar} H_{ba} e^{i\omega t} C_a \]

Can solve exactly for \( H_{ab} = \text{constant pulse} \)

Use RWA for \( H_{ab} = \cos\omega t \)

Both give similar results

But for general \( H_{ab}(t) \) cannot solve

\( \Rightarrow \) turn to perturbation theory

Basic assumption: \( H' \) has small effect on system

\( \Rightarrow C_a \approx C_b \) don't change much

Assume particle starts in \( |a\rangle \)

\( C_a(0) = 1 \quad C_b(0) = 0 \)

Approximate evolution as follows

Zeroth order:

No effect of \( H' \), so

\( C_a^{(0)}(t) = 1 \quad C_b^{(0)}(t) = 0 \)

Here \( C_a^{(n)} \approx n^{th} \) order approx to \( C_a \)
First order:

\[ C_b = -\frac{i}{k} H_{ba} e^{i\omega_0 t} C_a \]

Integrate:

\[ C_b(t) = -\frac{i}{k} \int_0^t H_{ba}(t') e^{i\omega_0 t'} dt' \]

Also:

\[ C_a(t) = -\frac{i}{k} H_{ab} e^{i\omega_0 t} C_b^{(0)} = 0 \]

So:

\[ C_a(t) = 1 \]

Second order:

Use \( C_a^{(1)} \) and \( C_b^{(1)} \) on RHS

\[ C_b^{(1)} = -\frac{i}{k} H_{ba} e^{i\omega_0 t} C_a^{(1)} \]

But:

\[ C_a^{(1)} = C_a^{(0)} \]

So:

\[ C_b^{(1)} = C_b^{(1)} \]

(i.e., no term \( \propto H_{ba}^2 \))

While:

\[ C_a^{(1)} = -\frac{i}{k} H_{ab} e^{-i\omega_0 t} C_b^{(1)} \]

\[ = -\frac{i}{k} H_{ab} e^{-i\omega_0 t} \left[ -\frac{i}{k} \int_0^t H_{ba}(t') e^{-i\omega_0 t'} dt' \right] \]

Integrate:

\[ C_a(t) = 1 - \frac{1}{i^2} \int_0^t H_{as}/t' e^{-i\omega_0 t'} \left[ \int_0^{t'} H_{ba}(t'') e^{i\omega_0 t''} dt'' \right] dt' \]
And so forth. Mostly just use 1st order result

valid when \( \int_0^T H_{eq} e^{i\omega_0't} dt' \) is small

Example: \( H_{eq} = t \sum \cos \omega t \)

\[
C_{b^{(1)}} = \left. -i \Delta \int_0^t \cos \omega_0 t' e^{i\omega_0 t'} dt' \right. \\
= -i \Delta \int_0^t \left( e^{i\omega_0 t'} + e^{-i\omega_0 t'} \right) dt' \\
= -i \frac{\Delta}{2} \left[ \frac{e^{i\omega_0 t} - 1}{\omega_0 + \omega} + \frac{e^{-i\omega_0 t} - 1}{\omega_0 - \omega} \right]_0^t \\
= -i \frac{\Delta}{2} \left[ \frac{e^{i\omega_0 t} - 1}{\omega_0 + \omega} + \frac{e^{-i\omega_0 t} - 1}{\omega_0 - \omega} \right] \\
\]

Simplifies if \( \omega \approx \omega_0 \) (like for RWA)

Then \( \frac{1}{\omega_0 + \omega} \ll \frac{1}{\omega_0 - \omega} \), drop 1st term

\[
C_{b^{(1)}}(t) \approx -i \frac{\Delta}{2} \int e^{-i\Delta t} \left( e^{-i\Delta t} - e^{i\Delta t} \right) \\
= -i \frac{\Delta}{2} e^{-i\Delta t/2} (e^{-i\Delta t/2} - e^{i\Delta t/2}) \\
= -i \frac{\Delta}{2} e^{-i\Delta t/2} (-2i \sin \frac{\Delta t}{2}) \\
C_{b^{(1)}}(t) = -i \frac{\Delta}{2} e^{-i\Delta t/2} \sin \frac{\Delta t}{2} \\
\]

Compare to non-perturbative result:

\[
c_b(t) = -i \frac{\Delta}{\Delta_0^2 - \omega^2} e^{-i\Delta t/2} \sin \frac{\Delta t}{2} \\
\]
For small $A$, have $\sqrt{\Delta + A^2} = A + \frac{\Delta^2}{2A} \approx A$, 1st order

So, $C_6(t) \rightarrow -\frac{i\Delta}{A} e^{i\Delta t/2} \sin \frac{\Delta t}{2} = C_6(0)(t)$, as we expect.

Also, note that if $\Delta \to 0$ have

$$C_6(t) \rightarrow -\frac{i\Delta}{A} \frac{\Delta t}{2} = -i \frac{\Delta^2 t}{2}$$

Amplitude increases linearly in time

Valid as long as $A^2 t \ll 1$

So $C_6$ doesn't change much

Get same from non-perturbative result:

for small $t$, $-\frac{i\Delta}{\sqrt{\Delta^2 + A^2}} e^{-i\Delta t/2} \sin \frac{\Delta t}{2} \approx -i \frac{\Delta^2 t}{2}$

So perturbative/approx works in two regimes:

either $A^2 \ll 1, A$ (so $H'$ is small)
or $t \ll A^2$ (so $H'$ doesn't have time to act)

Both reflect general principle that PT works as long as $C_s$ don't change much.

In either case, get transition probability:

$$P_{asb}(t) = \frac{\Delta^2}{A^2} \sin^2 \frac{\Delta^2 t}{2}$$
That is for 2-level system

Generalize to arbitrary # of states
Problem 9.15, Yuehaw

Problem does case $H'=0$ const
for case $H'_{MN} = V_{MN} \cos \omega t$

replace $E_N - E_M \rightarrow E_N - E_M \pm \hbar \omega$

(whichever is smaller)

and $H'_{NN} \rightarrow \frac{1}{2} V_{NN}$

so

$$
\rho_{NM} = |V_{MN}|^2 \frac{\sin^2 \left[ \frac{(E_N - E_M + \hbar \omega)t}{2\hbar} \right]}{(E_N - E_M \pm \hbar \omega)^2}
$$