Lecture 2

Last time, introduced idea of composing two spin-½ particles in a total composite spin

For example, total spin of hydrogen atom

Find four possible results:

$S = 1$:

$|1\,1\rangle = |\uparrow\uparrow\rangle$

$|1\,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

$|1\,-1\rangle = |\downarrow\downarrow\rangle$

$S = 0$:

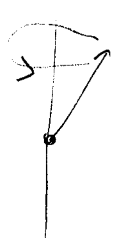
$|0\,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

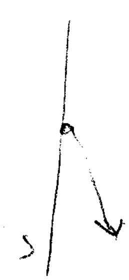
In most real systems, $S=0$ & $S=1$ states have different energies. Often referred to as "singlet" and "triplet" states.

So how can we understand this result? Can draw some pictures.

Represent spin as a classical vector. Has definite z-component, but not x or y. So picture vector as precessing around z-axis

So for spin-½, draw

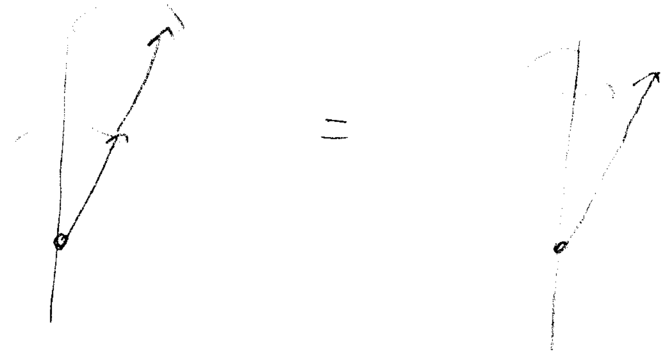
$|\uparrow\rangle \rightarrow$ 

$|\downarrow\rangle \rightarrow$ 

Note, can't take this too seriously, its more of a mnemonic. But a useful one, even so.

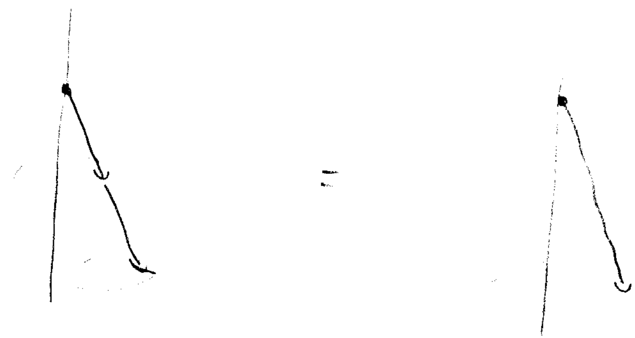
Then add spins by adding vectors:

$|\uparrow\uparrow\rangle =$



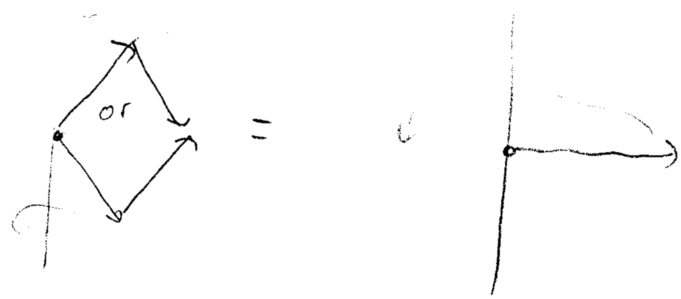
Big spin up

$|\downarrow\downarrow\rangle =$



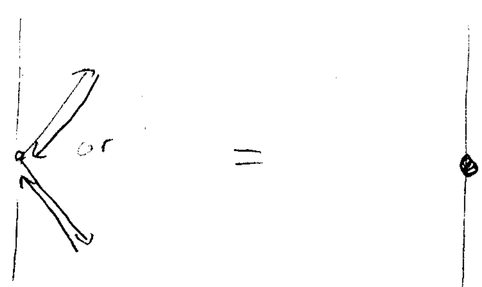
Big spin down

Think of $|10\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$ as precessing in phase:



big spin, $m=0$

and $|00\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$ as precessing out of phase:

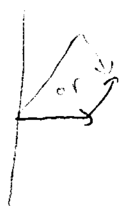


spin zero

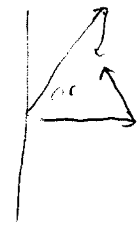
We can generalize this to more complex combinations.

For instance, when combining a spin 1 x spin $\frac{1}{2}$ system, get six states:

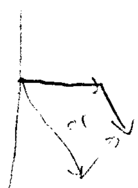
: $|J=\frac{3}{2}, m=\frac{3}{2}\rangle$



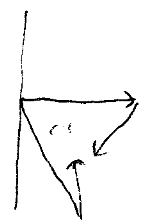
$|J=\frac{3}{2}, m=\frac{1}{2}\rangle$



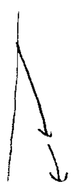
$|J=\frac{1}{2}, m=\frac{1}{2}\rangle$



$|J=\frac{3}{2}, m=-\frac{1}{2}\rangle$



$|J=\frac{1}{2}, m=-\frac{1}{2}\rangle$



$|J=\frac{3}{2}, m=-\frac{3}{2}\rangle$

General result: When two systems with ang. momenta J_1 and J_2 are combined, result has ang. momentum J ranging from $|J_1-J_2|$ to J_1+J_2 in integer steps.

ie, for spins $J_1=\frac{1}{2}$, $J_2=\frac{1}{2}$, so $J=0$ or $\underline{1}$

for $J_1=1$, $J_2=\frac{1}{2}$, get $J=1-\frac{1}{2}=\frac{1}{2}$ or $1+\frac{1}{2}=\frac{3}{2}$

for $J_1=3$, $J_2=2$, get $J=1,2,3,4$ or 5

I won't prove, but I'll check one thing for you: number of states should stay the same.

Number should stay the same, doesn't matter how you count them.

For $J_1 = 3$, have $2J_1 + 1 = 7$ states
$J_2 = 2$, have $2J_2 + 1 = 5$ states
so $7 \times 5 = 35$ total

For composite state have

$J = 1$	3 states
$J = 2$	5
$J = 3$	7
$J = 4$	9
$J = 5$	11
sum =	35 as required.

General proof isn't too hard, but not really relevant to QM... I'll post on web site.

Note that what we're really doing here is changing our basis.

Uncoupled basis is $\{ |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle \}$

or generally $\{ |J_1 m_1 J_2 m_2\rangle \}$

Coupled basis is $\{ |11\rangle, |10\rangle, |1{-1}\rangle, |00\rangle \}$

or generally $\{ |J\, m\rangle \}$

Like always, you can work in whatever basis is more convenient.

Uncoupled basis is good for effects that treat each particle separately.

Coupled basis is good for effects that interact with total angular momentum.

We'll see examples of both as we go.

We know that changing basis is just a linear transformation

Can always write

$$|J\,m\rangle = \sum_i C_i\,|J_1\,m_1\,J_2\,m_2\rangle$$

C_i = coefficients.

depend on $J\,m$, $J_1\,m_1$, $J_2\,m_2$. Only m_1 & m_2 vary.

So write

$$|J_m\rangle = \sum_{m_1 m_2} C^{J_1 J_2 J}_{m_1 m_2 m}\,|J_1 m_1\,J_2 m_2\rangle$$

C's called Clebsch-Gordon coefficients

For instance, we know $C^{\frac{1}{2}\,\frac{1}{2}\,1}_{\frac{1}{2}\,\frac{1}{2}\,1} = 1 \quad : \quad |1\,1\rangle = |\uparrow\uparrow\rangle$

$$C^{\frac{1}{2}\,\frac{1}{2}\,1}_{\frac{1}{2}\,-\frac{1}{2}\,0} = C^{\frac{1}{2}\,\frac{1}{2}\,1}_{-\frac{1}{2}\,\frac{1}{2}\,0} = \frac{1}{\sqrt{2}} \quad : \quad |1\,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$C^{\frac{1}{2}\,\frac{1}{2}\,0}_{\frac{1}{2}\,-\frac{1}{2}\,0} = -C^{\frac{1}{2}\,\frac{1}{2}\,0}_{-\frac{1}{2}\,\frac{1}{2}\,0} = \frac{1}{\sqrt{2}} \quad : \quad |0\,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

and $C^{\frac{1}{2}\,\frac{1}{2}\,1}_{-\frac{1}{2}\,-\frac{1}{2}\,-1} = 1$ $\qquad |1\,-1\rangle = |\downarrow\downarrow\rangle$

Note that $C = 0$ unless $m_1 + m_2 = m$
and $|J_1 - J_2| \le J \le J_1 + J_2$

No simple formula for C's, but values listed in special table, page 188 in book.

To read: find $J_1 \times J_2$ table

Numbers on top give J, m
Numbers on sides give m_1, m_2

Numbers in middle are C-G coeffs, with implied $\sqrt{\ }$

IE, if table says $-\frac{1}{2}$, means $-\frac{1}{\sqrt{2}}$

For instance 2×1 table says

$$\overset{J\ m}{|3\ 0\rangle} = \frac{1}{\sqrt{5}}\,\overset{m_1\,m_2}{|1\,-1\rangle} + \sqrt{\tfrac{3}{5}}\,|0\,0\rangle + \sqrt{\tfrac{1}{5}}\,|-1\ 1\rangle$$

Use same coeffs to go from coupled to uncoupled.

$\frac{3}{2} \times 1$ table gives

$$\overset{m_1\ m_2}{|\tfrac{1}{2}\ 0\rangle} = \sqrt{\tfrac{3}{5}}\,\overset{J\ m}{|\tfrac{5}{2}\ \tfrac{1}{2}\rangle} + \sqrt{\tfrac{1}{15}}\,|\tfrac{3}{2}\ \tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

Probably seems a bit abstract right now, but we'll apply this a few times later in course, give more concrete examples.