Lecture 18

Exams: You can use computer or tables for integrals.

Last time, looked at Zeeman effect.

Weak field, \( B_{\text{external}} \ll B_{\text{internal}} \)

Get \( E^{(i)} = \mu_B g_J m_J B_{\text{ext}} \)

Looks like particle w/spin \( J \), moment \( g_J \mu_B \)

Other easy case: strong field \( B_{\text{ext}} \gg B_{\text{int}} \)

Then use \( |lm, sm_s\rangle \) basis, get

\[
E^{(s)}_{\text{Zeeman}} = \langle \frac{e}{2m} (L^2 + 2S^2) \beta^2 \rangle = \mu_B B (m + 2ms)
\]

Then apply FS as perturbation to that.

No more degeneracy, so just need

\[
\langle l m_e s m_s | H_{\text{FS}} | l m_e s m_s \rangle
\]

Have \( H_{\text{FS}} = H_{\text{rel}} + H_{\text{so}} \)

\[
-\frac{p^4}{8m^3c^3} - \frac{e^2}{2 \frac{1}{4\pi\varepsilon_0}} \frac{1}{m^2c^2r^3} \hat{S} \cdot \hat{L}
\]

Already did \( H_{\text{rel}} \) in \( 0s \) basis, no change

\[
\langle H_{\text{rel}} \rangle = -\frac{\langle E \rangle^2}{2mc^2} \left[ \frac{4\pi}{k_{\frac{1}{2}}} - 3 \right]
\]
For spin orbit, need \( \langle m_2 m_3 | S \cdot L | m_2 m_3 \rangle = \langle S_x L_x + S_y L_y + S_z L_z \rangle \)

But \( \langle S_x L_x \rangle = \langle m_s | S_x | m_s \rangle \langle m_x | L_x | m_x \rangle = 0 \)

same for \( y \)

Leaves \( \langle S_+ L_+ \rangle = \pm \hbar m_2 \)

and \( \langle H'_{SO} \rangle = \frac{1}{2} \frac{e^2}{4 \pi \hbar \epsilon_0} \frac{\hbar^2}{m^2 c^2} \frac{m_2 m_3}{l(l+\frac{3}{2})(l+1) n^3 a^3} \)

All together,

\[
\begin{align*}
E_{FS}^{(l)} &= -\frac{E_n}{n^3} \alpha^2 \left( \frac{3}{4} \right) - \left[ \frac{l(l+1) - m_2 m_3}{l(l+\frac{3}{2})(l+1) n^3 a^3} \right]^3 
\end{align*}
\]

You get to explore in HW 6.23

Hard case is when Zeeman, FS are comparable

Need to treat \( H' = H'_z + H'_{FS} \) together

Since \( H'_z \) & \( H'_{FS} \) have different symmetries, can't easily find basis where \( H' \) is diagonal

Comes back to fundamental problem of degenerate PT: manifold of degenerate states, need to diagonalize \( H' \) on manifold
Can't do for general $n$ ... need to treat each manifold separately

Ground state is trivial (no spin-orbit)
so let's do $n=2$

Step 1: Pick a basis:
If use $|j m_j\rangle$, $H'_{FS}$ is diagonal
$|m_m\rangle \Rightarrow H'_{FS}$ is diagonal

Either works, but $H'_{FS}$ is more complicated,
pick $|j m_j\rangle$

Step 2: Construct matrix $H'$

For $FS$, have $\langle j m_j | H'_{FS} | j m_j \rangle = \frac{E^2}{2mc^2} \frac{1}{n^4} \left( 3 - \frac{4n}{j+\frac{1}{2}} \right)$

$= \frac{E^2}{2mc^2} \left( 3 - \frac{8}{j+\frac{1}{2}} \right)$

Have $j = \frac{1}{2}$ or $\frac{3}{2}$

$j = \frac{1}{2} \Rightarrow \langle H'_{FS} \rangle = -\frac{5}{32} \frac{E^2}{mc^2} \equiv -5 \chi$

$j = \frac{3}{2} \Rightarrow \langle H'_{FS} \rangle = -\frac{1}{32} \frac{E^2}{mc^2} \equiv -\chi$

$\chi = \frac{1}{32} \frac{E^2}{mc^2}$
1/2 is helicity. Need to express \( |jm\rangle \) states in \( |lm\rangle \) basis to evaluate \( S_{12} \).

Use Clebsch–Gordan coefficients:

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Use to get:

\[
\langle j'm'|H_{1/2}|jm\rangle = \sum_{m'_{12}} \beta_{m'm_{12}} \langle j'm'|S_1 \cdot S_2 |jm\rangle
\]

So:

\[
\langle j'm'|H_{1/2}|jm\rangle = \frac{eB}{2m} \langle j'm'\mid S_1 \cdot S_2 \mid jm \rangle
\]
\[
\begin{align*}
\langle 3 \mid 3 \rangle &= \langle 3 \mid 3 \rangle \\
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\end{align*}
\]
\[
- \frac{\sqrt{2}}{3} \langle \frac{1}{2} \ | \ 1 S_{\pm} \ | \ 1 \frac{1}{2} \rangle + \frac{\sqrt{2}}{3} \langle \frac{1}{2} \ | \ -1 S_{\pm} \ | \ 1 \frac{1}{2} \rangle = - \frac{\sqrt{2}}{3} \chi
\]

Same procedure for all the rest

Get \[ \langle y_i \ | H' \ | y_j \rangle = \begin{bmatrix}
5y - \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5y + \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 8 - 2\beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 8 + 2\beta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 - \frac{2\sqrt{2}}{3}\beta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5y - \frac{2\sqrt{2}}{3}\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 8 + \frac{2\sqrt{2}}{3}\beta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5y + \frac{2\sqrt{2}}{3}\beta & 0
\end{bmatrix}
\]

where \[ \beta = \mu_0 B \]

Can see it is mostly diagonal:

\[ H' \] only couples states with same \( l \) and same \( m_j \)

So was very smart to put those states next to each other in basis set

\[ 2Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8 \]

Just leaves 2x2 blocks to diagonalize

Finish next time.