Lecture 1

- Go over syllabus

Section 4.4.3 Last semester, finished with discussion of spin.
Pick up from there.

Recall electrons have $S = \frac{1}{2}$
Two spin states $|m = +\frac{1}{2}\rangle \equiv |\uparrow\rangle$
$|m = -\frac{1}{2}\rangle \equiv |\downarrow\rangle$

General state $|\psi\rangle = \cos\theta|\uparrow\rangle + e^{i\phi}\sin\theta|\downarrow\rangle$

Convenient to use raising and lowering operators.

Generally, $J_\pm |j\, m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\,|j, m\pm 1\rangle$

So for $J = S = \frac{1}{2}$, have

Also:

$S_z|\uparrow\rangle = \frac{\hbar}{2}$ $\quad S_+|\uparrow\rangle = 0$ $\quad S_+|\downarrow\rangle = \hbar|\uparrow\rangle$

$S_z|\downarrow\rangle = -\frac{\hbar}{2}$ $\quad S_-|\uparrow\rangle = \hbar|\downarrow\rangle$ $\quad S_-|\downarrow\rangle = 0$

What we're going to be talking about is how QM works with multiple particles

We'll start with spin. Let's say we have two spin-$\frac{1}{2}$ particles... say electron and proton of a hydrogen atom.

How do we write state of entire atom?

Now four states: $\{|\uparrow\uparrow\rangle \quad |\uparrow\downarrow\rangle \quad |\downarrow\uparrow\rangle \quad |\downarrow\downarrow\rangle\}$ make basis

where $|\uparrow\downarrow\rangle$ = electron $\uparrow$, proton $\downarrow$

But we'd like to know about total angular momentum

$$\vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)}$$

adding operators is nothing special, just apply each one to where it belongs.

For instance, if $|\psi\rangle = |m_1 \, m_2\rangle = |m_1\rangle |m_2\rangle$

then $S_z|\psi\rangle = \left(S_z^{(1)} + S_z^{(2)}\right)|m_1\rangle|m_2\rangle$

$$= |m_2\rangle S_z^{(1)}|m_1\rangle + |m_1\rangle S_z^{(2)}|m_2\rangle$$

$$= \hbar(m_1 + m_2)|m_1 \, m_2\rangle$$

Since our basis states have definite values of m_1 & m_2, they have definite values of S_z:

$|\uparrow\uparrow\rangle \rightarrow m = 1$
$|\uparrow\downarrow\rangle \rightarrow m = 0$
$|\downarrow\uparrow\rangle \rightarrow m = 0$
$|\downarrow\downarrow\rangle \rightarrow m = -1$

What does this say about total spin of atom?

We know $m = 1$ shows up for $L = 1$ systems

Suggest total spin $S = 1$, make sense since $\frac{1}{2} + \frac{1}{2} = 1$

But, $S=1$ system has three states: $m=-1, 0, 1$

We have an extra $m=0$. What's that about?

We can check for sure. Start with $|\uparrow\uparrow\rangle$ state,

Is this an eigenstate of S^2?

Use $S^2 = (\vec{S}_1 + \vec{S}_2)^2 = S_1^2 + S_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$

and $\vec{S}_1 \cdot \vec{S}_2 = S_x^{(1)} S_x^{(2)} + S_y^{(1)} S_y^{(2)} + S_z^{(1)} S_z^{(2)}$

Remember $S_x = \frac{1}{2}(S_+ + S_-)$

$S_y = \frac{1}{2i}(S_+ - S_-)$

So $S_x^{(1)} S_x^{(2)} = \frac{1}{4}\left(S_+^{(1)} S_+^{(2)} + S_-^{(1)} S_-^{(2)} + S_+^{(1)} S_-^{(2)} + S_-^{(1)} S_+^{(2)} \right)$

$S_y^{(1)} S_y^{(2)} = -\frac{1}{4}\left(S_+^{(1)} S_+^{(2)} + S_-^{(1)} S_-^{(2)} - S_+^{(1)} S_-^{(2)} - S_-^{(1)} S_+^{(2)} \right)$

and $\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(S_+^{(1)} S_-^{(2)} + S_-^{(1)} S_+^{(2)} \right) + S_z^{(1)} S_z^{(2)}$

So, apply S^2 to $|\uparrow\uparrow\rangle$: $S_1^2 |\uparrow\uparrow\rangle = (S_1^2 |\uparrow\rangle)|\uparrow\rangle = \hbar^2 (S_1)(S_1+1)|\uparrow\uparrow\rangle$

$= \hbar^2 \frac{1}{2} \cdot \frac{3}{2} |\uparrow\uparrow\rangle$

$= \frac{3}{4}\hbar^2 |\uparrow\uparrow\rangle$

$S_2^2 |\uparrow\uparrow\rangle = \frac{3}{4}\hbar^2 |\uparrow\uparrow\rangle$

$2\vec{S}_1 \cdot \vec{S}_2 |\uparrow\uparrow\rangle = 0 + 2 S_z^{(1)} S_z^{(2)} |\uparrow\uparrow\rangle = 2\hbar^2 \cdot \frac{1}{2} \cdot \frac{1}{2} |\uparrow\uparrow\rangle = \frac{1}{2}\hbar^2 |\uparrow\uparrow\rangle$

When $S_+^{(i)}$ hits $\uparrow\uparrow$, get zero

Thus $S^2|\uparrow\uparrow\rangle = \hbar^2\left(\frac{3}{4}+\frac{3}{4}+\frac{1}{2}\right)|\uparrow\uparrow\rangle$

$$= 2\hbar^2|\uparrow\uparrow\rangle = \hbar^2(1)(1+1)|\uparrow\uparrow\rangle$$

thus $|\uparrow\uparrow\rangle$ has total spin $= 1$

and $m = 1$, so write $|\uparrow\uparrow\rangle = |1\,1\rangle$

To sort out what's going on with $m=0$ states, just apply S_- to $|\uparrow\uparrow\rangle$

Know $S_-|1\,1\rangle = \hbar\sqrt{2}\,|1\,0\rangle$

But also $S_-|\uparrow\uparrow\rangle = \left(S_-^{(1)} + S_-^{(2)}\right)|\uparrow\uparrow\rangle$

$$= \left(S_-^{(1)}|\uparrow\rangle\right)|\uparrow\rangle + |\uparrow\rangle\left(S_-^{(2)}|\uparrow\rangle\right)$$

$$= \hbar\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right)$$

indicating that $|1\,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\uparrow\downarrow\rangle\right)$

symmetric combination of our basis states.

Easy to see that $|\downarrow\downarrow\rangle = |1\,{-1}\rangle$

So, we do get the three $S=1$ states.

But we have four states total. Remaining orthogonal one is $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

Evidently, this is not an $S=1$ state.
See what it is:

Apply $S^2 = S_1^2 + S_2^2 + 2S_z^{(1)}S_z^{(2)} + S_+^{(1)}S_-^{(2)} + S_-^{(1)}S_+^{(2)}$

$$S_1^2|\psi\rangle = S_2^2|\psi\rangle = \frac{3}{4}\hbar^2|\psi\rangle$$

$$S_z^{(1)}S_z^{(2)}|\psi\rangle = \frac{1}{\sqrt{2}}\left(S_z^{(1)}|\uparrow\rangle S_z^{(2)}|\downarrow\rangle - S_z^{(1)}|\downarrow\rangle S_z^{(2)}|\uparrow\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(-\frac{\hbar^2}{4}|\uparrow\downarrow\rangle - \left(-\frac{\hbar^2}{4}\right)|\downarrow\uparrow\rangle\right)$$

$$= -\frac{\hbar^2}{4}|\psi\rangle$$

$$S_+^{(1)}S_-^{(2)}|\psi\rangle = \frac{1}{\sqrt{2}}\left(S_+^{(1)}|\uparrow\rangle S_-^{(2)}|\downarrow\rangle - S_+^{(1)}|\downarrow\rangle S_-^{(2)}|\uparrow\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(0 - \hbar^2|\uparrow\downarrow\rangle\right) = -\frac{\hbar^2}{\sqrt{2}}|\uparrow\downarrow\rangle$$

$$S_-^{(1)}S_+^{(2)}|\psi\rangle = \frac{1}{\sqrt{2}}\left(S_-^{(1)}|\uparrow\rangle S_+^{(2)}|\downarrow\rangle - S_-^{(1)}|\downarrow\rangle S_+^{(2)}|\uparrow\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\hbar^2|\downarrow\uparrow\rangle - 0\right) = \frac{\hbar^2}{\sqrt{2}}|\downarrow\uparrow\rangle$$

So $$S^2|\psi\rangle = \left(\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 - \frac{1}{4}\hbar^2\right)|\psi\rangle + \frac{\hbar^2}{\sqrt{2}}\left(-|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$= \hbar^2|\psi\rangle \qquad - \hbar^2|\psi\rangle$$

$$= 0$$

Note that $0 = \hbar^2(0)(0+1)$, so $|\psi\rangle$ is eigenstate with $S=0$ (and $m=0$)

So in fact, there are four possible spin states for a hydrogen atom. Could have $S=1$, $m=0,\pm1$ or $S=0$, $m=0$