Lecture 3 - Impedance & Transfer Functions

Recall voltage divider:

\[ \frac{V_{in}}{V_{out}} = \frac{Z_1}{Z_1 + Z_2} \]

\[ V_{out} = V_{in} \cdot \frac{Z_1}{Z_2} \]

This assumes no current flows from the output: \( I_{out} = 0 \)

What if \( I_{out} \neq 0 \)?

Can solve:

\[ \frac{V_{in}}{I_{out}} = \frac{V_{out}}{I_{1}} \]

\[ I_1 = I_{out} + I_2 \]

\[ V_{in} = V_{out} - I_1 Z_1 \]

\[ \Rightarrow V_{in} - V_{out} = (I_{out} + \frac{V_{out}}{Z_2}) Z_1 \]

\[ V_{out} = I_2 Z_2 \]

\[ V_{in} - I_{out} Z_1 = V_{out} \left( 1 + \frac{Z_1}{Z_2} \right) \]

\[ V_{out} = \frac{Z_2}{Z_1 + Z_2} V_{in} - I_{out} \frac{Z_1 Z_2}{Z_1 + Z_2} \]

So as \( I_{out} \) increases, \( V_{out} \) decreases.

This is a generic effect. Can describe with another version of Thevenin's Theorem.

Any network of sources and impedances acts like a single source \( V_{eff} \) in series with single impedance \( Z_{out} \)

Here:

\[ \frac{V_{in}}{V_{out}} = \frac{Z_1}{Z_1 + Z_2} \]

\[ \Rightarrow V_{eff} = \frac{Z_1 Z_2}{Z_1 + Z_2} V_{out} \]
Equivalent circuit is easy to analyze:

\[ V_{\text{out}} = V_{\text{eff}} - I_{\text{out}} Z_{\text{out}} \]

Compare to above, see:

\[ V_{\text{eff}} = U_{1,2} \cdot \frac{Z_k^2}{Z_1 + Z_k} \]

\[ Z_{\text{out}} = \frac{Z_k^2}{Z_1 + Z_k} = Z_1 Z_2 \]

This is useful because it applies to any circuit

\[ V_1 \xrightarrow{R} V_{\text{out}} \xrightarrow{\frac{1}{R}} V_2 \xrightarrow{\frac{1}{R}} V_1 \]

\[ \Rightarrow V_{\text{out}} = V_{\text{eff}} - Z_{\text{out}} U_{\text{out}} \]

So for any circuit, output voltage decreases with current

\[ Z_{\text{out}} = -\frac{dV_{\text{out}}}{dI_{\text{out}}} = \text{"output impedance"} \]

If you know \( Z_{\text{out}} \), can say how much current can be supplied by a circuit.

Good power supply has low \( Z_{\text{out}} \)

How to determine \( Z_{\text{out}} \):

1) Assume some \( I_{\text{out}} \), and solve for \( V_{\text{out}} \), like we did with divider.

2) Easier: Assume \( I_{\text{out}} = 0 \) and solve for \( V_{\text{out}} = V_{\text{eff}} \) (then assume \( V_{\text{out}} = 0 \) and solve for \( I_{\text{out}} / \text{short} \)).

Then \( Z_{\text{out}} = V_{\text{eff}} / I_{\text{out}} \text{(short)} \)

Example:

\[ V_{\text{in}} \xrightarrow{\frac{Z_k}{Z_1}} I_{\text{short}} \]

\[ I_{\text{short}} = \frac{V_{\text{in}}}{Z_k} \]

\[ \Rightarrow Z_{\text{out}} = \frac{(V_{\text{in}} - \frac{Z_k Z_{\text{load}}}{Z_1})}{V_{\text{in}} / Z_1} = \frac{Z_k Z_{\text{load}}}{Z_1 + Z_k} \]
3) Good way to measure:
Measure $V_{\text{eff}} = V_{\text{out}}$ using voltmeter
Hook output to ground through impedance $Z_L$
measure $V_{\text{out}}$ again

Effective circuit: $V_{\text{eff}} \frac{Z_{\text{out}}}{Z_{\text{out}} + Z_L} V_{\text{out}}$

See $V_{\text{out}} = \frac{Z_L}{Z_{\text{out}} + Z_L} V_{\text{eff}}$

Solve for $Z_{\text{out}} = Z_L \times \frac{V_{\text{out}}}{V_{\text{eff}} - V_{\text{out}}}$

Some notes:
- Really only applies to linear circuits. Many circuits are nonlinear, but can still approximate as linear for (relatively) small $I_{\text{out}}$
- If $I_{\text{out}}$ is large, circuit dissipates internal power $I_{\text{out}}^2 R_{\text{out}}$
  If too large, circuit can break! Try not to short outputs to ground unless you know circuit can survive.
Related topic: Input Impedance

Output of any circuit acts like: \[ V_{\text{eff}} - \frac{Z_{\text{in}}}{Z_{\text{in}} + Z_{\text{out}}} V_{\text{out}} \]

Input of any circuit acts like: \[ U_{\text{i}} - \frac{Z_{\text{i}}}{Z_{\text{i}} + U_{\text{float}}} \]

\( U_{\text{float}} \) = voltage present at input if you don't hook anything up. (Generally \( \neq V_{\text{eff}/\text{out}} \))

Very often, \( U_{\text{float}} = \text{ground} \)

Example: \[ U_{\text{i}} - \frac{Z_{\text{i}}}{Z_{\text{i}} + U_{\text{out}}} \]

With no \( U_{\text{i}} \) source, have \( U_{\text{i}} = U_{\text{out}} = 0 \)
\[ \Rightarrow U_{\text{float}} = 0 \]

With \( U_{\text{i}} \), have \[ I_{\text{i}} = \frac{U_{\text{i}}}{Z_{\text{i}} + Z_{\text{L}}} \]
\[ \Rightarrow Z_{\text{i}} = Z_{\text{i}} + Z_{\text{L}} \]

Note we assume \( I_{\text{out}} = 0 \), because no load is present.
If we had a load \( Z_{\text{L}} \), \( Z_{\text{i}} \) would change.

Input impedance important for measuring instruments

 Voltmeter: \[ U_{\text{i}} - \frac{Z_{\text{i}}}{Z_{\text{i}} + U_{\text{out}}} \]

Describes fact that some current flows in to device, can affect measurement:

\[ \frac{U_{\text{V}}}{I_{\text{V}}} \]

\[ \frac{Z_{\text{i}}}{Z_{\text{i}} + R} \]

Forms divider, actually measures \[ V_{\text{meas}} = 9V \times \frac{Z_{\text{i}}}{Z_{\text{i}} + R} \]

Ideally, \( Z_{\text{i}} \gg R \), so \( V_{\text{meas}} \approx 9V \)
Or in ammeter:

\[ I \rightarrow \frac{V}{Z_{in}} \rightarrow \]

Needs some input voltage \( V_{in} = I Z_{in} \) to operate.

For instance:

\[ qV \rightarrow \frac{qV}{2R} \rightarrow \frac{qV}{2R + 2Z_{in}} \]

Unless \( Z_{in} \ll R \), I chooses when you add ammeter to circuit.

Introduce one more topic, discuss more next time:

transfer function

Often have relation \( V_{out} = ( ) V_{in} \)

divider: \( V_{out} = \frac{Z_{2}}{Z_{1} + Z_{2}} V_{in} \)

In general, write \( V_{out} = G V_{in} \)

\( G = G(w) \) depends on frequency

Generally complex

Call \( G = \) transfer function

Describes what a circuit does

Often useful to express \( G \) in polar form \( G = |G| e^{i\phi} \)

Show using two plots: \(|G| \) vs \( w \)

\( \phi \) vs \( w \)

But \( |G| \) and \( w \) typically vary a lot: use log plots

\( \log |G| \) vs \( \log w \) (or \( \log \phi \))

\( \phi \) vs \( \log w \)
Call pair: Bode plot

Convenient way to describe how circuit functions

\( \log |G| \) so convenient, it has a special unit! dB

Write \( g \text{ (in dB)} = 20 \log |G| \)

So 20 dB \( \Rightarrow \) \( |G| = 10 \)

-20 dB \( \Rightarrow \) \( |G| = 0.1 \)

Usually see Bode plot as \( g \) vs. \( \log_{10} f \)

\( \phi \text{ (deg) vs. } \log_{10} f \)