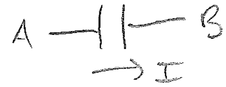
Lecture 2 - Complex Impedence

Last time, introduced capacitors



$$\Delta V = V_A - V_B = \frac{1}{C}\int I\,dt$$

$\Rightarrow$ differential eqn relating V & I

Simplifies in particular case of harmonic signals

$$V, I \propto \cos(\omega t + \phi)$$

- Note $\omega \sim \text{rad/s}$ - $\omega = 2\pi f$ $f \sim \text{Hz}$

Convenient to use complex representation

$$V(t) = \text{Re}\, V e^{i\omega t}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\hookrightarrow$ complex $= |V| e^{i\phi}$

Assume you know complex numbers

so $V(t) = |V| \,\text{Re}\, e^{i(\omega t + \phi)} = |V| \cos(\omega t + \phi)$

$|V| =$ amplitude

$\phi =$ phase

Similarly, $I(t) = \text{Re}\, I e^{i\omega t}$

Do calculations with complex form, take real part at end

- What if actual signal isn't harmonic?

Use Fourier transform $V(t) = \frac{1}{2\pi}\int V(\omega)\, e^{i\omega t}\, d\omega$

Any reasonable function = sum of harmonic fns

For linear circuits, treat each component separately, add up in end.

Apply to resistors & capacitors

Resistors simple: $V(t) = I(t) R$

$$V e^{i\omega t} = I e^{i\omega t} R$$

$\boxed{V = IR}$ $\rightarrow$ now a complex relation!

Capacitors

$$V(t) = \frac{1}{C}\int I(t)\,dt$$

$$Ve^{i\omega t} = \frac{1}{C}\int I e^{i\omega t}\,dt$$

$$= \frac{1}{C}\, I\, \frac{e^{i\omega t}}{i\omega}$$

$$\boxed{V = \frac{1}{i\omega C} I} \Rightarrow \quad |V| = \frac{|I|}{\omega C} \qquad \arg V = \arg I - 90^\circ$$

Same form as Ohm's Law: $V = (\ )\, I$

Here $(\ )$ = complex, freq-dependent

but math doesn't care

Generalize Ohm's Law: $V = IZ$

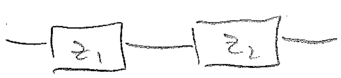
Z = impedance

$\mathrm{Re}\, Z$ = resistance

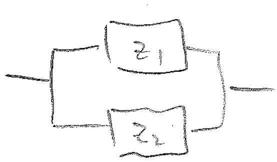
$\mathrm{Im}\, Z$ = reactance

Resistor has $Z = R$ Capacitor has $Z = \frac{1}{i\omega C}$

With this method, can solve any circuit like resistor network

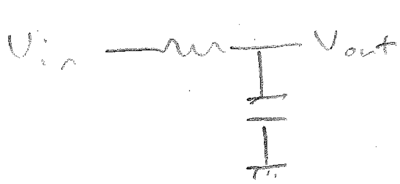


$\Rightarrow \quad Z_{TOT} = Z_1 + Z_2$

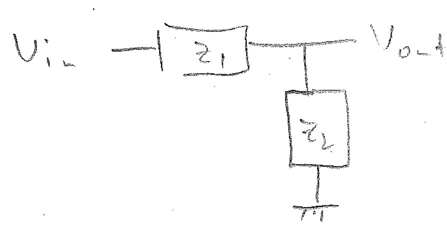


$\Rightarrow \quad Z_{TOT} = Z_1 \parallel Z_2 = \frac{Z_1 Z_2}{Z_1 + Z_2}$

Example:



$\Rightarrow$



Like voltage divider, solved last time

Use result: $V_{out} = \frac{Z_2}{Z_1 + Z_2} V_{in}$

Here $V_{out} = \frac{\frac{1}{i\omega C}}{R + \frac{1}{i\omega C}} V'_{in}$

$$= \frac{1}{1 + i\omega RC} V_{in}$$

Amplitude $|V_{out}| = \left| \frac{V_{in}}{1 + i\omega RC} \right| = \frac{|V_{in}|}{\sqrt{1+\omega^2 R^2 C^2}}$

~constant for $\omega \ll \frac{1}{RC}$

decreases for $\omega \gg \frac{1}{RC}$

Functions as low pass filter: attenuates high freq components

Also get phase shift

$$\frac{1}{1+i\omega RC} = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}} e^{i\phi}$$ (polar form $z = |z| e^{i\theta}$)

To get ϕ:

$$1 + i\omega RC = \sqrt{1+\omega^2 R^2 C^2}\, e^{-i\phi}$$

$$= \sqrt{1+\omega^2 R^2 C^2}\,(\cos\phi - i\sin\phi)$$

so $\cos\phi - i\sin\phi = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}} + i \frac{\omega RC}{\sqrt{1+\omega^2 R^2 C^2}}$

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = -\frac{\omega RC/\cancel{\sqrt{}}}{1/\sqrt{}} = -\omega RC$$

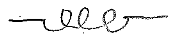
$$\phi = -\tan^{-1}(\omega RC)$$

For $\omega \ll \frac{1}{RC}$, $\phi \approx 0$

$\omega \gg \frac{1}{RC}$, $\phi \approx -90^\circ$

All together: $V_{out}(t) = \frac{1}{\sqrt{1+\omega^2 RC}}\, e^{i(\omega t + \phi_{in} - \tan^{-1}\omega RC)}$

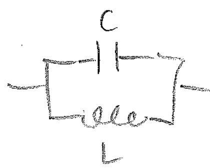
Discuss a few more components:

- Inductor 

  Has $V = L \frac{dI}{dt}$

  Impedance $Z = i\omega L$

  So for example:



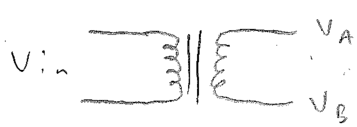
$$: \quad Z_{TOT} = Z_1 \parallel Z_2$$
$$= \frac{(i\omega L)\left(\frac{1}{i\omega C}\right)}{i\omega L + \frac{1}{i\omega C}}$$
$$= \frac{i\omega L}{1 - \omega^2 LC}$$

- Battery

  Generates voltage $V_A - V_B = V_{battery}$

  Like Ohm's law, only voltage difference is <u>defined</u>
  Absolute voltage of either terminal determined by rest of circuit ... "floating"

- Transformer



  For ac (oscillating) signals:

  $$V_A - V_B = k V_{in} \cos \omega t$$

  k can be larger, smaller, or equal to <u>1</u>

  Useful to change amplitude

  Also, output floats, like a battery