

# Lie Algebras: Abstract Theory of Weights

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These notes are based on chapter 13 in 'Introduction to Lie Algebras and Representation Theory', by James E. Humphreys.

Let  $\Phi$  be a root system in an euclidean space  $E$ , with Weyl group  $\mathcal{W}$ .

Recall, a subset  $\Phi$  of the euclidean space  $E$  is called a **root system** in  $E$  if the following axioms are satisfied:

(R1)  $\Phi$  is finite, spans  $E$  and it does not contain 0.

(R2) If  $\alpha \in \Phi$ , the only multiples of  $\alpha$  in  $\Phi$  are  $\pm\alpha$ .

(R3) If  $\alpha \in \Phi$ , the reflexion  $\sigma_\alpha$  leaves  $\Phi$  invariant.

(R4) If  $\alpha, \beta \in \Phi$  then  $\langle \beta, \alpha \rangle \in \mathbb{Z}$ .

For each  $\alpha \in E$  we define  $\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$ . Let  $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ .

**Lemma 1:**  $\Phi^\vee$  is a root system.

*Proof.* I will check the four axioms of the root system.

(R1) Since  $|\Phi^\vee| = |\Phi|$ , then  $\Phi^\vee$  is finite. Since  $\Phi$  spans  $E$  and  $\alpha^\vee$  is a non-zero scalar multiple of  $\alpha$  for all  $\alpha \in \Phi$ , then  $\Phi^\vee$  spans  $E$  and  $0 \notin \Phi^\vee$ .

(R2) Since  $(-\alpha)^\vee = \frac{2(-\alpha)}{((- \alpha), (- \alpha))} = -\frac{2\alpha}{(\alpha, \alpha)} = -\alpha^\vee$ , then the only multiples of  $\alpha^\vee$  are  $\pm\alpha^\vee$ .

(R3) I need to show that if  $\alpha^\vee \in \Phi^\vee$  then the reflection  $\sigma_{\alpha^\vee}$  leaves  $\Phi^\vee$  invariant. Let  $\alpha$  in  $\Phi$ , then for all  $\beta \in \Phi$ ,  $\sigma_\alpha(\beta) \in \Phi$ . Let  $\beta \in \Phi$ .

$$\begin{aligned}\sigma_{\alpha^\vee}(\beta^\vee) &= \beta^\vee - \frac{2(\beta^\vee, \alpha^\vee)}{(\alpha^\vee, \alpha^\vee)}\alpha^\vee \\ &= \frac{2\beta}{(\beta, \beta)} - \frac{2\left(\frac{2\beta}{(\beta, \beta)}, \frac{2\alpha}{(\alpha, \alpha)}\right)}{\left(\frac{2\alpha}{(\alpha, \alpha)}, \frac{2\alpha}{(\alpha, \alpha)}\right)} \frac{2\alpha}{(\alpha, \alpha)} \\ &= \frac{2\beta}{(\beta, \beta)} - \frac{4(\beta, \alpha)\alpha}{(\beta, \beta)(\alpha, \alpha)} \\ &= \frac{2}{(\beta, \beta)} \left( \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \right) \\ &= \frac{2}{(\beta, \beta)}\sigma_\alpha(\beta)\end{aligned}$$

But,

$$\begin{aligned}
(\sigma_\alpha(\beta), \sigma_\alpha(\beta)) &= \left( \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha, \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \right) \\
&= (\beta, \beta) - \frac{2(\beta, \alpha)(\beta, \alpha)}{(\alpha, \alpha)} - \frac{2(\beta, \alpha)(\alpha, \beta)}{(\alpha, \alpha)} + \frac{4(\beta, \alpha)^2(\alpha, \alpha)}{(\alpha, \alpha)^2} \\
&= (\beta, \beta) - \frac{4(\beta, \alpha)^2}{(\alpha, \alpha)} + \frac{4(\beta, \alpha)^2}{(\alpha, \alpha)} \\
&= (\beta, \beta)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sigma_{\alpha^\vee}(\beta^\vee) &= \frac{2}{(\beta, \beta)}\sigma_\alpha(\beta) \\
&= \frac{2}{(\sigma_\alpha(\beta), \sigma_\alpha(\beta))}\sigma_\alpha(\beta) \\
&= (\sigma_\alpha(\beta))^\vee
\end{aligned}$$

Thus,  $\sigma_{\alpha^\vee}(\beta^\vee) \in \Phi^\vee$ .

(R4) I need to show that  $\langle \alpha^\vee, \beta^\vee \rangle \in \mathbb{Z}$ . We have:

$$\begin{aligned}
\langle \alpha^\vee, \beta^\vee \rangle &= \frac{2(\beta^\vee, \alpha^\vee)}{(\alpha^\vee, \alpha^\vee)} \\
&= \frac{2\left(\frac{2\beta}{(\beta, \beta)}, \frac{2\alpha}{(\alpha, \alpha)}\right)}{\left(\frac{2\alpha}{(\alpha, \alpha)}, \frac{2\alpha}{(\alpha, \alpha)}\right)} \\
&= \frac{2(\beta, \alpha)}{(\beta, \beta)} \\
&= \frac{2(\alpha, \beta)}{(\beta, \beta)} \\
&= \langle \beta, \alpha \rangle
\end{aligned}$$

Since  $\Phi$  is a root system, then  $\langle \beta, \alpha \rangle \in \mathbb{Z}$ . Therefore  $\langle \alpha^\vee, \beta^\vee \rangle \in \mathbb{Z}$ .  $\square$

**Lemma 2:** If  $\Delta = \{\alpha_1, \dots, \alpha_l\}$  is a basis of  $E$  contained in  $\Phi$  such that for all  $\alpha \in \Phi$ ,

$\alpha = \sum_{i=1}^l k_i \alpha_i$  all  $k_i$  are nonnegative or all  $k_i$  are nonpositive, then  $\Delta$  is a base of  $E$ .

*Proof. Step 1:* Find  $\gamma$  regular such that  $(\gamma, \alpha_i) > 0$  for all  $1 \leq i \leq l$ .

For each  $1 \leq i \leq l$ , let  $P_i$  be the hyperplane generated by  $\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_l$ . Let  $\delta_i$  be the projection of  $\alpha_i$  on the orthogonal complement of  $P_i$ . I will show that  $(\delta_i, \alpha_i) > 0$ . Let  $v_1, \dots, v_{l-1}$  be an orthonormal basis of  $P_i$ . Then  $v_1, \dots, v_{l-1}, w$  where  $w = \frac{\delta_i}{|\delta_i|}$  is an orthonormal basis of  $E$ . Let  $v = \alpha_i$ . By Pythagoras theorem,  $(v, v) =$

$\sum_{i=0}^{l-1} (v, v_i)^2 + (v, w)^2$ , therefore  $(v, v) - \sum_{i=0}^{l-1} (v, v_i)^2 \geq 0$ . But,  $(\delta_i, \alpha_i) = |\delta_i|(w, v) = |\delta_i|(v - \sum_{j=1}^{l-1} (v, v_j)v_j, v) = |\delta_i| \left( (v, v) - \sum_{j=1}^{l-1} (v, v_j)^2 \right) \geq 0$ . Since  $\alpha_i \notin P_i$ , then  $(\delta_i, \alpha_i) > 0$ .

Let  $\gamma = \sum_{i=1}^l r_i \delta_i$  where  $r_i > 0$ . Since  $(\delta_i, \alpha_j) = 0$  for all  $i \neq j$ , then  $(\gamma, \alpha_i) = r_i (\delta_i, \alpha_i) > 0$ .

Let  $\alpha \in \Phi$ . Then  $\alpha = \sum_{i=1}^l k_i \alpha_i$  all  $k_i$  are nonnegative or all  $k_i$  are nonpositive.

Thus  $(\gamma, \alpha) = \sum_{i=1}^l k_i (\gamma, \alpha_i)$ . Since not all  $k_i$  are zero and  $(\gamma, \alpha_i) > 0$  and all  $k_i$  are nonnegative or all  $k_i$  are nonpositive, then  $(\gamma, \alpha) \neq 0$ . Therefore  $\gamma \notin P_\alpha$ . Since  $\alpha$  was arbitrary, then  $\gamma \notin \cup_{\alpha \in \Phi} P_\alpha$ . By the definition,  $\gamma$  is regular.

*Step 2:*  $\Delta$  is a base of  $\Phi$ .

We know from a theorem in 10.1, that the set  $\Delta(\gamma)$  of indecomposable elements of  $\Phi^+(\gamma) = \{\alpha \in \Phi : (\gamma, \alpha) > 0\}$  is a base of  $\Phi$ . Thus, it is enough to show that the elements of  $\Delta$  are indecomposable. Assume  $\alpha_i \in \Delta$  is decomposable, that is

$\alpha_i = \beta_1 + \beta_2$ , where  $\beta_1, \beta_2 \in \Phi^+(\gamma)$ . Since  $\Delta$  is a basis of  $E$ , then  $\beta_1 = \sum_{j=1}^l k_j \alpha_j$

and  $\beta_2 = \sum_{j=1}^l l_j \alpha_j$ . Since  $\Delta$  is a basis, then  $k_j, l_j \in \mathbb{Z}$  for all  $j$  and  $1 = k_i + l_i$  and

$k_j + l_j = 0$  for all  $j \neq i$ . We have two cases:

Case 1:  $k_j = l_j = 0$  for all  $i \neq j$

Then  $\beta_1 = k_i \alpha_i$  and  $\beta_2 = l_i \alpha_i$ . Since  $\Phi$  is a root system,  $k_i, l_i \in \{\pm 1\}$ . Then  $k_i + l_i \in \{-2, 0, 2\}$ . This contradicts  $k_i + l_i = 1$ .

Case 2:  $k_j \neq 0$  for some  $j \neq i$

Since either all  $k_i$  nonnegative or all  $k_i$  nonpositive, and similarly for  $l_i$ , then then one set must be nonnegative and one nonpositive. Without loss of generality, assume

$k_i \geq 0$  and  $l_i \leq 0$ . Then  $(\gamma, \beta_2) = \sum_{i=1}^l l_i (\gamma, \alpha_i)$ . Since  $\gamma$  is regular, then  $(\gamma, \alpha_i) > 0$

and  $(\gamma, \beta_2) > 0$ , we reached a contradiction.

Thus  $\alpha_i$  is indecomposable. Therefore  $\Delta = \Delta(\gamma)$ .  $\square$

**Corollary:** If  $\Delta$  is a base of  $\Phi$ , then  $\Delta^\vee$  is a base of  $\Phi^\vee$ .

*Proof.* Since  $(,)$  is positive definite, then  $(\alpha, \alpha) > 0$  for all  $\alpha \in E$ . Since  $\Delta$  is a base of  $\Phi$ , then  $\alpha = \sum_{i=1}^l k_i \alpha_i$  all  $k_i$  are nonnegative or all  $k_i$  are nonpositive. Then

$\alpha^\vee = \sum_{i=1}^l k'_i \alpha_i^\vee$  where  $k'_i = \frac{(\alpha_i, \alpha_i)}{(\alpha, \alpha)} k_i$  are all nonnegative or all nonpositive. By lemma 2,  $\Delta^\vee$  is a base of  $\Phi^\vee$ .

A **weight**  $\lambda$  is an element of the euclidean space  $E$  such that  $\langle \lambda, \alpha \rangle \in \mathbb{Z}$  for all  $\alpha \in \Phi$ . Let  $\Lambda$  denote the set of all weights. Since  $\langle, \rangle$  is linear in the first factor, then  $\Lambda$  is a subgroup of  $E$ . By the axiom (R4) of the definition of the root system,  $\Phi \subset \Lambda$ .  $\square$

**Lemma 3:** Let  $\Phi$  be a root system in an euclidean space  $E$ , with base  $\Delta$ . Let  $\Lambda$  be the set of weights. Then  $\lambda \in \Lambda$  iff  $\langle \lambda, \alpha \rangle \in \mathbb{Z}$  for all  $\alpha \in \Delta$ .

*Proof.* The forward implication follows from the definition of weight. Conversely, let  $\alpha \in \Phi$ . Since  $\langle \lambda, \alpha \rangle = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} = (\lambda, \frac{2\alpha}{(\alpha, \alpha)}) = (\lambda, \alpha^\vee)$ , then we need to show that

$(\lambda, \alpha^\vee) \in \mathbb{Z}$ . By the previous corollary,  $\Delta^\vee$  is a base of  $\Phi^\vee$ , therefore  $\alpha = \sum_{i=1}^l k_i \alpha_i^\vee$

where  $k_i \in \mathbb{Z}$  for all  $1 \leq i \leq l$ . Thus  $(\lambda, \alpha^\vee) = \sum_{i=1}^l k_i (\lambda, \alpha_i^\vee)$ . By assumption  $\langle \lambda, \alpha_i \rangle \in \mathbb{Z}$  for all  $1 \leq i \leq l$ , thus  $(\lambda, \alpha_i^\vee) \in \mathbb{Z}$ . Thus  $(\lambda, \alpha^\vee) \in \mathbb{Z}$  as a sum of integers.  $\square$

The **root lattice**  $\Lambda_r$  is the subgroup of  $\Lambda$  generated by  $\Phi$ .

**Lemma 4:**  $\Lambda_r$  is a lattice.

*Proof.* Let  $\Delta = \{\alpha_1, \dots, \alpha_l\}$  be a base of  $\Phi$ . Then  $\Delta$  is a basis of  $E$ . I will show that  $\Lambda_r$  is the  $\mathbb{Z}$ -span of  $\Delta$ .

Let  $\sum_{i=1}^l k_i \alpha_i$  be in the  $\mathbb{Z}$ -span of  $\Delta$ . By the axiom (R4) of the root system,  $\langle \alpha_i, \alpha_j \rangle \in \mathbb{Z}$ .

Thus  $\langle \sum_{i=1}^l k_i \alpha_i, \alpha_j \rangle = \sum_{i=1}^l k_i \langle \alpha_i, \alpha_j \rangle \in \mathbb{Z}$  as the sum of integers. Therefore,

by lemma 3,  $\sum_{i=1}^l k_i \alpha_i$  is a weight, and it is in the subgroup generated by  $\Delta \subset \Phi$ .

Let  $\lambda \in \Lambda_r$ . Then  $\lambda = \sum_{\alpha \in \Phi} k_\alpha \alpha$  where  $k_\alpha \in \mathbb{Z}$  for all  $\alpha \in \Phi$ . Since  $\Delta$  is a base

of  $\Phi$ , then  $\alpha = \sum_{i=1}^l k_{\alpha, i} \alpha_i$  where all  $k_{\alpha, i}$  are integers. Therefore  $\lambda = \sum_{i=1}^l k_i \alpha$ , where

$k_i = \sum_{\alpha \in \Phi} k_{\alpha, i} k_\alpha$ . Since  $\Phi$  is finite and all  $k_{\alpha, i}$  are integers, then  $k_i \in \mathbb{Z}$  for all  $1 \leq i \leq l$ .

Therefore  $\lambda$  is in the  $\mathbb{Z}$ -span of  $\Delta$ .  $\square$

For a fixed base  $\Delta$  of the root system  $\Phi$ , a weight  $\lambda$  is called **dominant** if  $\langle \lambda, \alpha \rangle \geq 0$  for all  $\alpha \in \Delta$ . A weight  $\lambda$  is called **strongly dominant** if  $\langle \lambda, \alpha \rangle > 0$  for all  $\alpha \in \Delta$ . Let  $\Lambda^+$  denote the set of dominant weights. By definition, the fundamental Weyl chamber relative to  $\Delta$ ,  $\mathfrak{C}(\Delta)$ , is the connected component of  $E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$  containing a regular  $\gamma$  such that  $\Delta = \Delta(\gamma)$  is the set of indecomposable elements of  $\Phi^+ = \{\alpha \in \Phi : (\gamma, \alpha) > 0\}$ . Therefore  $\Lambda^+$  is the set of weights lying in the closure of the fundamental Weyl chamber, and the set of strongly dominant weights is the intersection of the fundamental Weyl chamber with  $\Lambda$ .

Let  $\Delta = \{\alpha_1, \dots, \alpha_l\}$ . Let the basis dual (relative to the inner product) to  $\Delta^\vee$  be denoted by  $\{\lambda_1, \dots, \lambda_l\}$ . Then  $(\lambda_i, \alpha_j^\vee) = \delta_{ij}$ . An element of  $\{\lambda_1, \dots, \lambda_l\}$  is called a **fundamental dominant weight**. Indeed  $\lambda_i$  is a weight, since  $\langle \lambda_i, \alpha_j \rangle = (\lambda_i, \alpha_j^\vee) = \delta_{ij} \in \mathbb{Z}_{\geq 0}$ .

Note that,  $\sigma_{\alpha_i}(\lambda_j) = \lambda_j - \langle \lambda_j, \alpha_i \rangle \alpha_i = \lambda_j - \delta_{ij} \alpha_i$ .

**Lemma 5:**  $\Lambda$  is a lattice with basis  $\Delta' = \{\lambda_1, \dots, \lambda_l\}$ . Furthermore,  $\lambda \in \Lambda^+$  iff  $m_i = \langle \lambda, \alpha_i \rangle \geq 0$ .

*Proof.* I will show that  $\Lambda$  is the  $\mathbb{Z}$ -span of  $\Delta'$ .

Let  $\sum_{i=1}^l k_i \lambda_i$  be in the  $\mathbb{Z}$ -span of  $\Delta'$ . By lemma 3, it is enough to show that  $\langle \sum_{i=1}^l k_i \lambda_i, \alpha_j \rangle \in \mathbb{Z}$ . We have  $\langle \sum_{i=1}^l k_i \lambda_i, \alpha_j \rangle = \sum_{i=1}^l k_i \langle \lambda_i, \alpha_j \rangle = \sum_{i=1}^l k_i \delta_{ij} = k_j \in \mathbb{Z}$ .

Let  $\lambda \in \Lambda$ . Then  $m_i = \langle \lambda, \alpha_i \rangle \in \mathbb{Z}$  for all  $1 \leq i \leq l$ . Then  $\langle \lambda - \sum_{i=1}^l m_i \lambda_i, \alpha_j \rangle = \langle \lambda, \alpha_j \rangle - \sum_{i=1}^l m_i \langle \lambda_i, \alpha_j \rangle = m_j - \sum_{i=1}^l m_i \delta_{ij} = 0$  for all  $1 \leq i \leq l$ . Then  $(\lambda - \sum_{i=1}^l m_i \lambda_i, \alpha_j) = 0$  for all  $1 \leq j \leq l$ . Thus  $\lambda = \sum_{i=1}^l m_i \lambda_i$ .

$\lambda \in \Lambda^+$  iff  $\langle \lambda, \alpha_i \rangle \geq 0$  for all  $1 \leq i \leq l$  iff  $m_i \geq 0$  for all  $1 \leq i \leq l$ .  $\square$

Examples:

(1) Calculate the fundamental dominant weights of  $A_1$ .

Let  $\Phi = \{\pm\alpha_1\}$  be a root system of  $A_1$  with base  $\{\alpha_1\}$ . Let  $\lambda_1 = k_1 \alpha_1$ . Since  $\langle \lambda_1, \alpha_1 \rangle = \delta_{11} = 1$ , then  $2k_1 = 1$ . Therefore  $\alpha_1 = 2\lambda_1$ .

(2) Calculate the fundamental dominant weights of  $A_2$ .

We know from previous chapters that the Cartan matrix of  $A_2$  is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Say  $\lambda_1 = k_1\alpha_1 + k_2\alpha_2$ . Then

$$\begin{cases} k_1 \langle \alpha_1, \alpha_1 \rangle + k_2 \langle \alpha_2, \alpha_1 \rangle = 1 \\ k_1 \langle \alpha_1, \alpha_2 \rangle + k_2 \langle \alpha_2, \alpha_2 \rangle = 0 \end{cases} \quad \begin{cases} 2k_1 - k_2 = 1 \\ -k_1 + 2k_2 = 0 \end{cases} \quad \begin{cases} k_1 = \frac{2}{3} \\ k_2 = \frac{1}{3} \end{cases}$$

Thus  $3\lambda_1 = 2\alpha_1 + \alpha_2$ . Similarly,  $3\lambda_2 = \alpha_1 + 2\alpha_2$ .

The group  $\Lambda/\Lambda_r$  is called **the fundamental group** of  $\Phi$ .

**Lemma 6:** The fundamental group of  $\Phi$  has finite order equal to the determinant of the Cartan matrix of  $\Phi$ .

*Proof.* First I will prove the following: Let  $M$  be a free  $\mathbb{Z}$ -module of rank  $l$  with basis  $\{y_1, \dots, y_l\}$  and  $L$  be a  $\mathbb{Z}$ -submodule of  $M$  of rank  $l$  with basis  $\{x_1, \dots, x_l\}$ . Let  $T$  denote the change of basis matrix from  $\{y_1, \dots, y_l\}$  to  $\{x_1, \dots, x_l\}$ . Then the order of the group  $M/L$  is  $|\det T|$ .

We know from Algebra II, theorem 7.1, that there exists a basis  $\{y'_1, \dots, y'_l\}$  of  $M$  and integers  $m_1, \dots, m_l$  such that  $\{m_1y'_1, \dots, m_ly'_l\}$  is a basis of  $L$ . Furthermore, the integers  $m_i$  are unique up to multiplication by units and  $m_1|m_2|\dots|m_l$ . Thus, w.l.o.g we can assume  $m_i > 0$ .

I will show next, that  $\sum_{i=1}^l c_i y'_i$  where  $0 \leq c_i \leq m_i - 1$  is a system of cosets representatives of  $M/L$ . By using the division algorithm, one can see that any element of  $M$  is in one of these cosets, thus these cosets cover  $M$ . Since  $0 \leq c_i \leq m_i - 1$ , then  $\sum_{i=1}^l c_i y'_i - \sum_{i=1}^l d_i y'_i \in L$  iff  $(c_i - d_i)|m_i$  iff  $c_i = d_i$ . Therefore, all the cosets above are distinct. Then  $|M/L| = [M : L] = m_1 \dots m_l$ .

Let  $A$  be the change of basis matrix from  $\{y_1, \dots, y_l\}$  to  $\{y'_1, \dots, y'_l\}$ ,  $B = \text{diag}(m_1, \dots, m_l)$  the change of basis matrix from  $\{y'_1, \dots, y'_l\}$  to  $\{m_1y'_1, \dots, m_ly'_l\}$ , and  $C$  be the change of basis matrix from  $\{m_1y'_1, \dots, m_ly'_l\}$  to  $\{x_1, \dots, x_l\}$ . Since  $A$  and  $C$  are invertible matrices with integer elements, then  $\det A, \det C \in \{\pm 1\}$ . Since  $T = ABC$ , then  $|\det T| = \det B = m_1 \dots m_l$ . Therefore  $|M/L| = |\det T|$ .

Say  $\alpha_i = \sum_{j=1}^l m_{ij} \lambda_j$ , where  $m_{ij} \in \mathbb{Z}$ . Then  $\langle \alpha_i, \alpha_k \rangle = \sum_{j=1}^l m_{ij} \langle \lambda_j, \alpha_k \rangle =$

$\sum_{j=1}^l m_{ij} \delta_{jk} = m_{ik}$ . Thus the change of basis matrix from  $\{\alpha_1, \dots, \alpha_l\}$  to  $\{\lambda_1, \dots, \lambda_l\}$  is the Cartan matrix of  $\Phi$ . Then  $|\Lambda/\Lambda_r|$  is the determinant of the Cartan matrix.  $\square$

Examples: (1) Calculate the determinant of the Cartan matrix of  $A_l$ .

We know from chapter 11, that the Cartan matrix of  $A_l$  is

$$A_l = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ & & & & \dots & & \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

To calculate its determinant, we add all the columns to the first one and we expand the minors along the first column:

$$\begin{aligned} & \begin{vmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ & & & & \dots & & \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ & & & & \dots & & \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 1 & 0 & 0 & 0 & \dots & -1 & 2 \end{vmatrix} \\ &= \det(A_{l-1}) + (-1)^{l+1} \begin{vmatrix} -1 & 0 & 0 & \dots & 0 & 0 \\ 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 2 & -1 \end{vmatrix} \\ &= \det(A_{l-1}) + (-1)^{l+1}(-1)^{l-1} = \det(A_{l-1}) + 1 \end{aligned}$$

We know that  $A_1 = (2)$ , thus  $\det(A_1) = 2$ . By induction, it follows that  $\det(A_l) = l + 1$ .

(2) Calculate the determinant of the Cartan matrix of  $B_l$ .

We know from chapter 11, that the Cartan matrix of  $B_l$  is

$$B_l = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

Then  $B_1 = (2)$  and thus  $\det(B_1) = 2$ . We have  $B_2 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$  and, thus,  $\det(B_2) = 2$ . To calculate the determinant we expand the minors along the first

column.

$$\begin{aligned}
B_l &= \begin{vmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & \dots & & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{vmatrix} \\
&= 2 \begin{vmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & \dots & & & & \\ 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{vmatrix} + (-1)(-1)^3 \begin{vmatrix} -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & \dots & & & & \\ 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{vmatrix} \\
&= 2\det(B_{l-1}) + (-1) \begin{vmatrix} 2 & -1 & \dots & 0 & 0 & 0 \\ & \dots & & & & \\ 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{vmatrix} \\
&= 2\det(B_{l-1}) - \det(B_{l-2})
\end{aligned}$$

By induction, one can check that  $\det(B_l) = 2$ .

(3) Calculate the determinant of the Cartan matrix of  $C_l$ .

We know from chapter 11, that the Cartan matrix of  $C_l$  is

$$C_l = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & \dots & & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -2 & 2 \end{pmatrix}$$

Then  $C_1 = (2)$  and thus  $\det(C_1) = 2$ . We have  $C_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$  and, thus,  $\det(C_2) = 2$ . A calculation similar to the one for  $B_l$  gives the same recurrence relation,  $\det(C_l) = 2\det(C_{l-1}) - \det(C_{l-2})$ . Similarly, it follows that  $\det(C_l) = 2$ .

(4) Calculate the determinant of the Cartan matrix of  $D_l$ .

We know from chapter 11, that the Cartan matrix of  $D_l$  is

$$D_l = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 & 0 & 0 \\ & & \dots & & & & & & \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

We have  $D_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$  and, thus,  $\det(D_2) = 2$ .

$$\det(D_3) = \begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = - \begin{vmatrix} -1 & -1 \\ 2 & 0 \end{vmatrix} + 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = -2 + 6 = 4$$

A calculation similar to the one for  $B_l$  gives the same recurrence relation,  $\det(D_l) = 2\det(D_{l-1}) - \det(D_{l-2})$ . Similarly, it follows that  $\det(D_l) = 4$ .

Note that the Weyl group  $\mathcal{W}$  leaves  $\Lambda$  invariant. Indeed if  $\lambda$  is a weight then  $\langle \lambda, \alpha \rangle \in \mathbb{Z}$  for all  $\alpha \in \Phi$ . We know that an element of the Weyl group preserves the inner product and it permutes the elements of  $\Phi$ . Thus  $\sigma\lambda$  has the desired property.

**Lemma 7:** Each weight is conjugate under  $\mathcal{W}$  to one and only one dominant weight. If  $\lambda$  is dominant, then  $\sigma\lambda \prec \lambda$  for all  $\sigma \in \mathcal{W}$ , and if  $\lambda$  is strongly dominant, then  $\sigma\lambda = \lambda$  only when  $\sigma = 1$ .

*Proof.* To prove this lemma we will use exercise 10.14 and lemma 10.3B.

Recall lemma 10.3B: Let  $\lambda, \mu \in \overline{\mathfrak{C}(\Delta)}$ . If  $\sigma\lambda = \mu$  for some  $\sigma \in \mathcal{W}$ , then  $\sigma$  is a product of simple reflections which fix  $\lambda$ ; in particular  $\lambda = \mu$ .

I will prove next exercise 10.14: prove that each point of  $E$  is  $\mathcal{W}$ -conjugate to a point in the closure of the fundamental base chamber relative to a base  $\Delta$ .

Recall  $\beta \prec 0$  if  $\beta = \sum_{i=1}^l k_i \alpha_i$  where all  $k_i \leq 0$  and  $\mu \prec \lambda$  iff  $\mu - \lambda \prec 0$ . Let  $\mu \in E$ . Let

$\sigma \in \mathcal{W}$  such that  $\lambda = \sigma\mu$  is maximal with the property that  $\mu \prec \lambda$ . Assume  $\lambda \notin \overline{\mathfrak{C}(\Delta)}$ . Then there exists  $\alpha_i \in \Delta$  such that  $\langle \lambda, \alpha_i \rangle < 0$ . Then  $\sigma_{\alpha_i} \lambda - \lambda = -\langle \lambda, \alpha_i \rangle \alpha_i$ . Thus  $\lambda \prec \sigma_{\alpha_i} \lambda$ . This contradicts the choice of  $\lambda$ . Therefore  $\lambda \in \overline{\mathfrak{C}(\Delta)}$ .

Let  $\lambda$  be a weight. By the exercise 10.14, there exists  $\lambda' \in \overline{\mathfrak{C}(\Delta)}$   $\mathcal{W}$ -conjugate to  $\lambda$ . Since  $\Lambda$  is closed under the action of the Weyl group, then  $\lambda'$  is also a weight, thus a dominant weight.

Assume  $\lambda$  is conjugate to two dominant weights,  $\lambda'$  and  $\lambda''$ . Then  $\lambda' \in \overline{\mathfrak{C}(\Delta)}$  and  $\lambda'' \in \overline{\mathfrak{C}(\Delta)}$ . By theorem 10.3B,  $\lambda' = \lambda''$ .

If  $\lambda$  is dominant, then  $\lambda \in \overline{\mathfrak{C}(\Delta)}$ . Let  $\mu \in \overline{\mathfrak{C}(\Delta)}$  be the  $\mathcal{W}$ -conjugate of  $\sigma\lambda$  by the reflection  $\tau$ . Then  $\lambda, \mu \in \overline{\mathfrak{C}(\Delta)}$  with  $\tau\sigma\lambda = \mu$ . By the above result  $\sigma\lambda \prec \mu$  and by lemma 10.3B  $\mu = \lambda$ . Therefore  $\sigma\lambda \prec \lambda$ .

Assume  $\lambda$  is strongly dominant and  $\sigma\lambda = \lambda$ . Then  $\lambda \in \overline{\mathfrak{C}(\Delta)}$ . From 10.1, we know that  $\Delta(\sigma\lambda) = \sigma\Delta(\lambda)$ , and thus  $\Delta = \sigma(\Delta)$ . By theorem 10.3e,  $\sigma = 1$ .  $\square$

Note that the following is possible:  $\mu$  is dominant,  $\lambda$  is not dominant and  $\mu \prec \lambda$ .

**Lemma 8:** Let  $\lambda \in \Lambda^+$ . Then the number of dominant weights  $\mu \prec \lambda$  is finite.

*Proof.* Let  $\mu$  be a dominant weight such that  $\mu \prec \lambda$ . Since  $\mu$  is a dominant weight, then  $\langle m\mu, \alpha \rangle \geq 0$  for all  $\alpha \in \Delta$ . Since  $\lambda \in \Lambda^+$ , then  $\langle \lambda, \alpha \rangle \geq 0$  for all  $\alpha \in \Delta$ .

Then  $\langle \lambda + \mu, \alpha \rangle \geq 0$  for all  $\alpha \in \Delta$ . Since  $\mu \prec \lambda$ , then  $\lambda - \mu = \sum_{i=1}^l k_i \alpha_i$  with  $k_i \geq 0$  for all  $1 \leq i \leq l$ . Thus  $(\lambda + \mu, \lambda - \mu) = \sum_{i=1}^l k_i (\lambda + \mu, \alpha_i) \geq 0$ . Therefore  $(\lambda, \lambda) - (\mu, \mu) \geq 0$ . Thus  $\mu \in \{(x, x) \in E : (x, x) \leq (\lambda, \lambda) \cap \Lambda^+\}$ . Since then set  $\{(x, x) \in E : (x, x) \leq (\lambda, \lambda)\}$  is compact and  $\Lambda^+$  is discrete, then their intersection is finite.  $\square$

**Lemma 9:** Let  $\delta = \frac{1}{2} \sum_{\alpha \succ 0} \alpha$ . Then  $\delta = \sum_{j=1}^l \lambda_j$ , so  $\delta$  is a strongly dominant weight.

*Proof.* Recall corollary to lemma 10.2B:  $\sigma_\alpha \delta = \delta - \alpha$  for all  $\alpha \in \Delta$ . Since  $\sigma_\alpha \delta = \delta - \langle \delta, \alpha \rangle \alpha$ , then  $\langle \delta, \alpha \rangle = 1$  for all  $\alpha \in \Delta$ . We showed already that  $\delta = \sum_{i=1}^l \langle \delta, \lambda_i \rangle \lambda_i$ . Thus  $\delta = \sum_{j=1}^l \lambda_j$ .

By lemma 5,  $\delta \in \Lambda^+$ .  $\square$