A Note on Cumulants and Möbius Inversion

Let $\Omega$ be a probability space equipped with a $\sigma$-algebra $\mathcal{A}$ and a probability measure $\nu$. For any $p \in [1, +\infty)$ we denote the corresponding real $L^p$ space by $L^p(\Omega)$. Let $\mathcal{E}$ be the intersection of these $L^p$ spaces. We work in this space, in order for arbitrary moments to be well defined. We will define a collection of multilinear forms $E_n^T$, $n \geq 1$, on this space $\mathcal{E}$:

$E_n^T : \mathcal{E}^n \to \mathbb{R}$

$(X_1, \ldots, X_n) \mapsto E_n^T[X_1, \ldots, X_n]$

This construction is by induction on $n$. One lets $E_1^T[X] = E[X]$, the usual expectation. Then for $n \geq 2$ one lets

$E_n^T[X_1, \ldots, X_n] = E[X_1 \cdots X_n] - \sum_{\pi \in \mathcal{L}_n, \pi \neq \hat{1}} \prod_{I \in \pi} E^T_{|I|}[(X_i)_{i \in I}]$

For this to make sense, one must show by the same induction that the multilinear forms $E_n^T$ are symmetric. This implies that an expression such as $E_n^T[\cdot]$ is well defined, and does not need the choice of an ordering for the collection of random variables indexed by $I$. The goal of this note is to use Möbius inversion in order to find an explicit expression for the cumulants in terms of the moments.

Note that we denoted by $\mathcal{L}_n$ the partition lattice on the set $[n] = \{1, 2, \ldots, n\}$. We also write $\hat{0}$ for the smallest element, i.e., the partition made of singletons, and we write $\hat{1}$ for the greatest element, i.e., the partition with one block equal to $[n]$ itself.

From the inductive definition of the cumulants one has the fundamental property that

$E[X_1 \cdots X_n] = \sum_{\pi \in \mathcal{L}_n} \prod_{I \in \pi} E^T_{|I|}[(X_i)_{i \in I}]$

For given $n$, and given choice of the possibly repeated random variables $X_1, \ldots, X_n$, this fundamental property also holds for subcollections. Namely for any $I \subseteq [n]$ one has

$E[\prod_{i \in I} X_i] = \sum_{\pi_I \in \mathcal{L}_I} \prod_{J \in \pi_I} E^T_{|J|}[(X_j)_{j \in J}]$ (1)

where $\mathcal{L}_I$ is the lattice of partitions of the set $I$. Now define, for any partition $\pi \in \mathcal{L}_n$

$f(\pi) = \prod_{I \in \pi} E[\prod_{i \in I} X_i]$

By (1) one has

$f(\pi) = \prod_{I \in \pi} \sum_{\pi_I \in \mathcal{L}_I} \prod_{J \in \pi_I} E^T_{|J|}[(X_j)_{j \in J}]$

then collecting the partitions $\pi_I$ into one partition $\pi'$ of $[n]$ which is finer than $\pi$, one gets

$f(\pi) = \sum_{\pi' \preceq \pi} g(\pi')$

1
with the new function
\[ g(\pi) = \prod_{I \in \pi} E^T_{|I|}(\{X_i\}_{i \in I}) \]

Then by Möbius inversion one has
\[ g(\pi) = \sum_{\pi' \leq \pi} \mu_{\mathcal{L}_n}(\pi', \pi) f(\pi') \]

In particular
\[ E^n_T[X_1, \ldots, X_n] = g(\hat{1}) = \sum_{\pi \in \mathcal{L}_n} \mu_{\mathcal{L}_n}(\pi, \hat{1}) \prod_{i \in \pi} E|I|X_i \]

Finally, in today’s lecture we have seen that
\[ \mu_{\mathcal{L}_n}(\pi, \hat{1}) = (-1)^{|\pi|-1}(|\pi| - 1)! \]

so we have an explicit formula for the cumulants. Of course \( E_T^2[X_1, X_2] = \text{cov}(X_1, X_2) \) and \( E_T^2[X, X] = \text{var}(X) \).