

Multivariable Power Series  
Reversion and Lagrange-Good  
Inversion via Quantum Field  
Theory

Abdelmalek Abdesselam  
Université Paris XIII

ArXiv : math.CO/0208173  
- math.CO/0208174

What is quantum field theory, really?

→ a generalization of calculus!

I - A game of symbolic integration

1) Composition

2) Reversion

3) Lagrange-Good Inversion

II - Feynman Diagrams: Like M. Jourdain  
physicists have been computing with  
combinatorial species without knowing.

# I. A Game of Symbolic Integration:

$K$  commutative field.

$F \in K[[X_1, \dots, X_n]] \Leftrightarrow$  function  $K^n \rightarrow K$

$F = (F_i)_{1 \leq i \leq n}$ ,  $F_i \in K[[X_1, \dots, X_n]] \Leftrightarrow$  function  $K^n \rightarrow K^n$

$u$  collection of variables  $u_1, \dots, u_n$   
 $du = du_1 \dots du_n$

$$u \cdot v = u_1 v_1 + \dots + u_n v_n$$

Rule 1:  $\int du F(u) \delta(u) = F(0)$

Rule 2:  $\int du e^{-uv} = \delta(v)$

Rule 3: All rules of calculus allowed:

Integration by parts, Fubini, change of variables (no 1.1 for Jacobian) ...

I-1) Composition:

$$F, G : K^n \rightarrow K^n \quad (\text{no constant term})$$

$$(F \circ G)_i(X) = \int d\bar{s} ds d\bar{t} dt d\bar{u} du \quad s_i \times$$

$$\exp(-\bar{s}s - \bar{t}t - \bar{u}u + \bar{s}F(t) + \bar{t}G(u) + \bar{u}X)$$

\*

"Proof":  $\int d\bar{u} e^{-\bar{u}u + \bar{u}X} = \delta(u - X)$  (Rule 2)

$$\int du \delta(u - X) \dots = \text{replace } u \text{ by } X \text{ in } \dots$$

(Rule 1 & 3)

$$\Rightarrow \text{RHS} = \int d\bar{s} ds d\bar{t} dt \quad s_i e^{-\bar{s}s - \bar{t}t + \bar{s}F(t) + \bar{t}G(X)}$$

$$= \int d\bar{s} ds \quad s_i e^{-\bar{s}s + \bar{s}F(G(X))} \quad (\text{same steps})$$

$$= F(G(X))_i$$

Comments: • (\*) is a combinatorial statement!

• generalization to  $F \circ F^2 \circ \dots \circ F^p$

immediate.

# I-2) Reversion:

Pb:  $F: K^n \rightarrow K^n$  with invertible linear term and no constant term.

?  $F^{-1}(y)$  the compositional inverse

## Computation:

let  $\Omega: K^n \rightarrow K$  (not necessarily without constant term)

$$\int d\bar{u} du \Omega(u) e^{-\bar{u}F(u) + \bar{u}y} = ?$$

$$\text{let } v = F(u) - y \Rightarrow u = F^{-1}(v+y)$$

$$du = \det[\partial F^{-1}(v+y)] dv$$

Notation: If  $F = (F_i(x))_{1 \leq i \leq n}$   $x = (x_1, \dots, x_n)$

$$\partial F(z) = \left( \frac{\partial F_i}{\partial x_j}(z) \right)_{1 \leq i, j \leq n}$$

the Jacobian matrix of  $F$  at the point  $z$ .

**Rule 3**

$$\Rightarrow \int d\bar{u} du \Omega(u) e^{-\bar{u}F(u) + \bar{u}y} = \int d\bar{v} dv \Omega(F^{-1}(v+y)) \det[\partial F^{-1}(v+y)] e^{-\bar{v}v}$$

**Rule 2**

$$= \int d\bar{v} dv \delta(v) \Omega(F^{-1}(v+y)) \det[\partial F^{-1}(v+y)]$$

**Rule 1**

$$= \Omega(F^{-1}(y)) \det[\partial F^{-1}(y)]$$

In particular

$$F^{-1}(y)_i = \frac{\int d\bar{u} du u_i e^{-\bar{u}F(u) + \bar{u}y}}{\int d\bar{u} du e^{-\bar{u}F(u) + \bar{u}y}}$$

### I-3) Lagrange - Good inversion:

(6)

Theorem:

Let  $G = (G_i)_{1 \leq i \leq n}$  be a system of formal power series in  $n$  letters

$F = (F_i)_{1 \leq i \leq n}$  in  $K[[X_1, \dots, X_n]]$  without constant term

defined by

$$F_i = X_i G_i(F)$$

then ...

Oops I forgot!

Computation:

let  $\Omega : K^n \rightarrow K$ .

$$\int d\bar{u} du \Omega(u) e^{-\bar{u}(u - XG(u))} = ?$$

equation satisfied by  $F$  ( $\dots = 0$ )

$$\bar{u} X G(u) := \sum_{i=1}^m \bar{u}_i X_i G_i(u_1, \dots, u_n)$$

We let  $H(u) = u - XG(u) \Rightarrow F = H^{-1}(0)$

change of variable  $v = H(u) \Rightarrow u = H^{-1}(v)$

and  $du = \det(\partial H^{-1}(v)) dv$

$$\Rightarrow \int d\bar{u} du \Omega(u) e^{-\bar{u}u + \bar{u}XG(u)} = \int d\bar{u} dv \Omega(H^{-1}(v)) \det(\partial H^{-1}(v)) e^{-\bar{u}v}$$

(Rule 3)

$$= \int dv \delta(v) \Omega(H^{-1}(v)) \det(\partial H^{-1}(v))$$

(Rule 2)

$$= \Omega(H^{-1}(0)) \det(\partial H^{-1}(0))$$

$$= \Omega(F) \frac{1}{\det(\partial H(H^{-1}(0)))}$$

or

$$\int d\bar{u} du \Omega(u) e^{-\bar{u}u + \bar{u}XG(u)} = \Omega(F) \frac{1}{\det(\text{Id} - X\partial G(F))}$$

Second way to do the computation:

$$\int d\bar{u} du \Omega(u) e^{-\bar{u}u + \bar{u}X G(u)} = \int d\bar{u} du \Omega(u) e^{-\bar{u}u} \sum_{M_1, \dots, M_n \geq 0} \frac{1}{M_1! \dots M_n!} (\bar{u}_1 X_1 G_1(u))^{M_1} \dots (\bar{u}_n X_n G_n(u))^{M_n}$$

$$= \sum_{M \in \mathbb{N}^n} \frac{X^M}{M!} \int d\bar{u} du \underbrace{\bar{u}^M e^{-\bar{u}u}}_{\left(-\frac{\partial}{\partial u}\right)^M e^{-\bar{u}u}} \Omega(u) G(u)^M$$

Multiindex notation

Integration by parts

$$= \sum_{M \in \mathbb{N}^n} \frac{X^M}{M!} \int d\bar{u} du e^{-\bar{u}u} \left(\frac{\partial}{\partial u}\right)^M [\Omega(u) G(u)^M]$$

$$= \sum_{M \in \mathbb{N}^n} \frac{X^M}{M!} \left(\frac{\partial}{\partial u}\right)^M_{u=0} [\Omega(u) G(u)^M]$$

Lagrange - Good inversion (implicit form)

$$\Omega(F) \frac{1}{\det(\delta_{ij} - X_i \partial_j G_i(F))} = \sum_{M \in \mathbb{N}^n} \frac{X^M}{M!} \left(\frac{\partial}{\partial u}\right)^M_{u=0} \Omega(u) G(u)^M$$

Remarks:

- $F_i(X) = \frac{\int d\bar{u} du u_i e^{-\bar{u}u + \bar{u}X G(u)}}{\int d\bar{u} du e^{-\bar{u}u + \bar{u}X G(u)}}$  Combinatorial

- Generalization to  $F_i = \sum_{j=1}^n X_{ij} G_j(F)$  easy.

# II Feynman (-Clifford-Sylvester) diagrams:

• Tensor calculus + generating functions

$$F : K^n \rightarrow K^n$$

component of degree  $d \in K^n \otimes \text{Sym}^d((K^n)^*)$

→ "matrix" element

$$f_{i, \alpha_1, \dots, \alpha_d} = \text{diagram of } F \text{ with } d \text{ inputs } \alpha_1, \dots, \alpha_d \text{ and output } i$$

*symmetric*

$$F_i(x) = \sum_{d \geq 1} \frac{1}{d!} \sum_{\alpha_1, \dots, \alpha_d=1}^n \text{diagram of } F \text{ with } d \text{ inputs } \alpha_1, \dots, \alpha_d \text{ and output } i \cdot x_{\alpha_1} \dots x_{\alpha_d}$$

NO MULTIINDICES

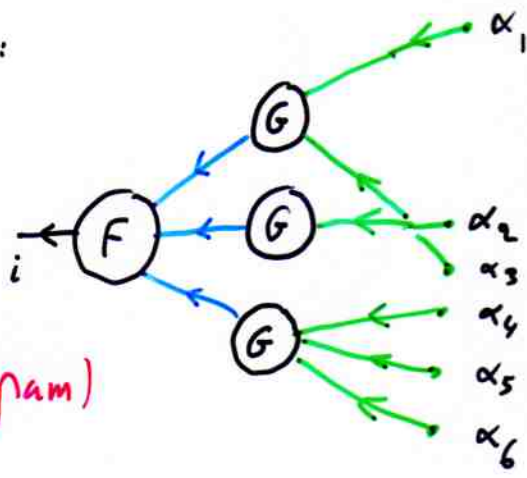
"Blob"  
of  
F

$$\text{diagram of } F \text{ with } d \text{ inputs } \alpha_1, \dots, \alpha_d \text{ and output } i = \frac{\partial}{\partial x_{\alpha_1}} \dots \frac{\partial}{\partial x_{\alpha_d}} \Big|_{x=0} F_i(x)$$

Knowing  $F$  = knowing the Blob of  $F$ .

Pb: Given  $\text{diagram of } F$  and  $\text{diagram of } G$  compute  $\text{diagram of } F \circ G$

Example:



$$(i, \alpha_1, \dots, \alpha_6 \in \{1, \dots, n\})$$

(Diagram)

$$\rightarrow \sum_{j_1, j_2, j_3=1}^m f_{i, j_1, j_2, j_3} g_{j_1, \alpha_1, \alpha_3} g_{j_2, \alpha_2} g_{j_3, \alpha_4, \alpha_5, \alpha_6}$$

(Amplitude)

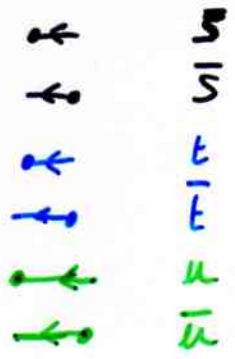
\* Need:

1) Encoding of the picture:

E finite set (of half lines) (#(E) = 20 above)

Feynman diagram structure on E = information to build the picture (except  $i, \alpha_1, \dots, \alpha_6$ )

e.g.: • specification of the type of half line :

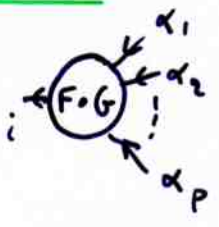


- labeling of external legs
- blob structure & Contraction

2) A rule associating an amplitude (in  $K$ ) to a Feynman diagram structure on  $E$ .

\* Remark:  $E \rightarrow \mathcal{M}(E) = \{ \text{Feyn. structure on } E \}$   
is a functor *combinatorial species*.

\* Result:



$$= \sum_{[E, F] \in \mathcal{M}(E)} \frac{1}{\#(\text{Aut}(E, F))}$$

*Classes*

*Amplitude*  
 $\mathcal{A}(E, F, i, \alpha_1, \dots, \alpha_p)$   
*indices = external data not included in  $E, F$ .*

$$= \int d\bar{s} ds d\bar{t} dt d\bar{u} du \quad s_i \bar{u}_{\alpha_1} \dots \bar{u}_{\alpha_p} \exp(-\bar{s}s - \bar{t}t - \bar{u}u + \bar{s}F(t) + \bar{t}G(u))$$

\* The same idea works for the composition of more than two "functions", reversion, Lagrange-Good and probably more.

Conclusion:

- \* Use Wick's theorem as a definition of formal Gaussian integration.
- \* Here complex bosonic fields (permanents)
- \* But also
  - real bosonic fields (hafnians)
  - complex fermionic fields (determinants)
  - real fermionic fields (Pfaffians)

\*

mixed situations → supersymmetry

? bijective Berezin change of variable formula