On the Ground State Energy for Massless Nelson Models

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I - Introduction
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I- Introduction :

Massless translation invariant Nelson model $(d \geq 3)$

$$\mathfrak{h} = L^2(\mathbb{R}^d)$$

Fock space $\quad \mathcal{F}(\mathfrak{h}) = \underset{\underset{\Omega}{\cup}}{\mathbb{C}} \bigoplus_{n=1}^{\infty} \mathfrak{h}^{\otimes_s^n}$ bosons

free vacuum

$a(k)$, $a^*(k)$

CCR :

$$[a(k), a(k')] = [a^*(k), a^*(k')] = 0$$

$$[a(k), a^*(k')] = \delta(k - k')$$

$$a(k)\Omega = 0$$

Dispersion relation $\quad \omega(k) = |k|$

$$H_f = \int_{\mathbb{R}^d} \omega(k)\, a^*(k) a(k)\, d^d k$$

$H_{-\frac{1}{2}}$ completion of $\mathcal{S}_{\mathbb{R}}(\mathbb{R}^d)$ for $\quad (f,g)_{-\frac{1}{2}} = \int_{\mathbb{R}^d} \frac{\overline{\hat{f}(k)}\, \hat{g}(k)}{2\omega(k)}\, d^d k$

$$\phi(f) = \int_{\mathbb{R}^d} \frac{d^d k}{\sqrt{2\omega(k)}} \left(\hat{f}(k) a(k) + \hat{f}(-k) a^*(k) \right) \qquad \text{for } f \in H_{-\frac{1}{2}}$$

Form factor $\rho \in H_{-\frac{1}{2}}$ such that $\frac{\hat{\rho}}{\omega} \in L^2$.

$\rho_x(\cdot) = \rho(\cdot + x)$

Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F}(h)$

Hamiltonian $H_\lambda = -\frac{1}{2}\Delta_x + H_f + \lambda\, \phi(\rho_x)$

coupling

Translation invariance:

H_λ commutes with total momentum $P = -i\nabla_x + P_f$

$$P_f = \int_{\mathbb{R}^d} k\, a^*(k)a(k)\, d^dk$$

$$\rightarrow \quad H_\lambda \simeq \int_{\mathbb{R}^d}^{\oplus} H_\lambda(P)\, d^dP$$

c-number

fiber Hamiltonian $$H_\lambda(P) = \frac{1}{2}(P - P_f)^2 + \lambda\, \phi(\rho_0) + H_f$$

on $\mathcal{F}(h)$.

s.a. for λ real, bounded below

$$\rightarrow \quad \underline{E_\lambda(P) = \inf \sigma(H_\lambda(P))}$$

Problem: Regularity of $E_\lambda(P)$ as a function of λ and P.

regularity in P $\rightarrow$ $$\frac{1}{m_{ren}} = \frac{\partial^2}{\partial P^2} E_\lambda(P) \Big|_{P=0}$$

renormalized mass of dressed particle

$\rightarrow$ construction of scattering states (Pizzo 2005)

Previous results:

- Fröhlich 1973 : a.e. differentiability in P
  $\nabla_P E_\lambda(P)$ locally Lipschitz

- Bach, Chen, Fröhlich, Sigal 2007
- Chen 2008
- Fröhlich, Pizzo 2010

  } : C^2 regularity in P for QED

- Griesemer, Hasler 2009 : analyticity in λ & P
  for less IR singular models

II - Theorem: (A.A. & D. Hasler 2010)

For ρ such that $\frac{\hat{\rho}}{\sqrt{\omega}}, \frac{\hat{\rho}}{\omega} \in L^2$ and $\hat{\rho} \geq 0$ a.e.

$E_\lambda(P)$ is jointly analytic in λ & P in domain

$$|\lambda| < \frac{1}{2}(1-|P|)^{-3/2} \times \left(\int_{\mathbb{R}^d} d^d k \, \frac{|\hat{\rho}(k)|^2}{|k|^2} \right)^{-1/2}$$

One has explicit convergent expression:

$$E_\lambda(P) = \frac{P^2}{2} - \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\lambda^2}{4}\right)^n \sum_{\substack{\tau \text{ tree} \\ \text{on } \{1,\dots,n\}}} \int_{[0,1]^\tau} \prod_{\{i,j\}\in\tau} dh_{\{i,j\}}$$

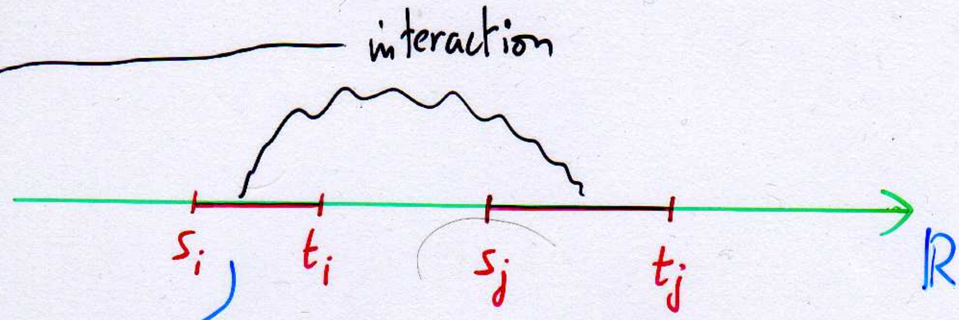
$$\int_{\mathbb{R}^{2n-1}} \prod_{j=2}^{n} ds_j \prod_{j=1}^{n} dt_j \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} d^d k_j \prod_{j=1}^{n} \left(e^{-|k_j||s_j - t_j| - (k_j \cdot P)(s_j - t_j)} \frac{|\hat{\rho}(k_j)|^2}{|k_j|} \right)$$

$$\times \prod_{\{i,j\}\in\tau} \left(-k_i \cdot k_j \, A_{ij}(s,t) \right)$$

$$\times \exp\left[-\frac{1}{2} \sum_{i=1}^{n} k_i^2 A_{ii}(s,t) - \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} k_i \cdot k_j \, A_{ij}(s,t) \, u(\tau,h)_{\{i,j\}} \right].$$

$s_1 := 0$ (Stat Mech Cluster Expansion)

Interacting gas of intervals $[s_j, t_j]$ on real line



$$\int_{\mathbb{R}^d} d^d k_i \; \frac{|\hat{\rho}(k_i)|^2}{|k_i|} \; e^{-|k_i||s_i - t_i| - (k_i \cdot P)(s_i - t_i)}$$

$$k_i \cdot k_j \; A_{ij}(s,t)$$

$$A_{ij}(s,t) = C(s_i, s_j) - C(s_i, t_j) - C(t_i, s_j) + C(t_i, t_j)$$

for $C(u,v) = \min(u,v)$ = covariance of 1d Brownian motion starting at 0.

matrix $\underline{A(s,t) \text{ is sym } \geq 0}$.