Notes on the Brydges-Kennedy-Abdesselam-Rivasseau forest interpolation formula

There are many instances in mathmatical physics where one tries to understand joint probability measures for a collection of random variables $X_1, \ldots, X_N$, with N large, of the form

$$e^{-\sum_{i=1}^{N} V(x_i)}\ d\mu_C(x)$$

where $d\mu_C$ is a Gaussian measure on $\mathbb{R}^N$. The dependence between these random variables is entierly due to the Gaussian measure which, in general, is given by covariances

$$C_{ij} = \mathrm{cov}(X_i, X_j)$$

which do not vanish for $i \neq j$. A typical procedure one uses in this type of problem is to try to interpolate between the given covariance matrix C and the covariance obtained by killing the off-diagonal entries. The outcome is what is called a cluster expansion in the constructive field theory literature. The first such expansions appeared in the context of the construction of the ϕ^4 model in 2 dimensions and in the infinite volume limit. They are due to Glimm, Jaffe and Spencer [11, 12, 10]. See also [21, 17] for a simpler presentation on the $\mathbb{Z}^d$ lattice, instead of the continuum model. Later on, a new simpler interpolation/expansion method was introduced for the same purpose by Brydges, Battle and Federbush [7, 4, 5]. The Brydges-Kennedy-Abdesselam-Rivasseau or BKAR formula offers yet another simpler interpolation formula. It allows the resummation of QFT Feynman diagrams according to spanning subtrees. It can also be used in order to prove the Penrose-Rota inequality for Mayer expansion coefficients. What follows is a self-contained presentation of the BKAR identity which is nothing more than a sophisticated form of the Fundamental Theorem of Calculus. The BKAR formula was dubbed the "constructive Swiss knife" in [19].

Let us consider a finite set $E \neq \emptyset$, and let us denote by $E^{(2)}$ the set unordered pairs $\{a, b\}$, where a and b are any distinct elements in E. Of course $|E^{(2)}| = \binom{|E|}{2}$. We will consider the space $\mathbb{R}^{E^{(2)}}$ of multiplets $s = (s_l)_{l \in E^{(2)}}$ indexed by pairs $l \in E^{(2)}$, and functions defined on a particular compact convex set $\mathcal{K}_E$ in this space. Let Π_E denote the set of partitions of E. For any partition $\pi = \{X_1, \ldots X_q\}$ in Π_E we associate a vector $v_\pi = (v_{\pi,l})_{l \in E^{(2)}}$ defined as

$$v_{\pi,l} = \mathbb{1}_{\{\exists i, 1 \leq i \leq q, l \subset X_i\}}\ .$$

Now $\mathcal{K}_E$ is by definition the convex hull of the vectors v_π, for $\pi \in \Pi_E$. It is easy to see that $\mathcal{K}_E$ affinely generates $\mathbb{R}^{E^{(2)}}$. Indeed, let $\hat{0}$ be the partition entierly made of singletons, and for any pair $l \in E^{(2)}$ let $\hat{l}$ denote the partition made of the two element set l and the singletons $\{a\}$, for $a \in E \backslash l$. Then, the vectors $v_{\hat{l}} - v_{\hat{0}}$, for $l \in E^{(2)}$ form a basis of the vector space $\mathbb{R}^{E^{(2)}}$. As a result, the open domain $\Omega_E = \mathring{\mathcal{K}}_E$ is nonempty, and $\mathcal{K}_E$ is equal to the closure $\bar{\Omega}_E$. Let $C^k(\bar{\Omega}_E)$ denote the usual space of functions of class C^k on the domain Ω_E

which, together with their derivatives up to order k, admit uniformly continuous extentions to the closure $\mathcal{K}_E = \bar{\Omega}_E$ (see, e.g., [3]). It is easy to see that for any f in $C^1(\bar{\Omega}_E)$ the Fundamental Theorem of Calculus

$$f(t) = f(s) + \int_0^1 du\ \frac{d}{du} f((1-u)t + us)$$

holds for any multiplets s and t in $\mathcal{K}_E$, even on the boundary. This will be used repeatedly in the following.

Now a simple graph with vertex set E can be thought of as a subset of the complete graph $E^{(2)}$. A forest $\mathfrak{F}$ is a graph with no circuits, and it is made of a vertex-disjoint collection of trees. Let $\mathfrak{F}$ be a forest, and let $\vec{h} = (h_l)_{l\in\mathfrak{F}}$ be a vector of real parameters indexed by the edges l in the forest $\mathfrak{F}$. To such data we canonically associate a multiplet $s(\mathfrak{F},\vec{h}) = (s(\mathfrak{F},\vec{h})_l)_{l\in E^{(2)}}$ in $\mathbb{R}^{E^{(2)}}$ as follows. Let a, and b be two distinct elements in E. If a and b belong to two distinct connected components of the forest $\mathfrak{F}$, then $s(\mathfrak{F},\vec{h})_{\{a,b\}} = 0$. Otherwise let, by definition, $s(\mathfrak{F},\vec{h})_{\{a,b\}} = \min_l h_l$ where l belongs to the unique path in the forest $\mathfrak{F}$ joining a to b. We are now ready to state the BKAR formula.

Theorem 1 *[8, 1] Let $f \in C^{|E|-1}(\bar{\Omega}_E)$, and let $1 \in \mathbb{R}^{E^{(2)}}$ denote the multiplet with all entries equal to one. This is also the same as $v_{\hat{1}}$ where $\hat{1}$ is the single block partition $\{E\}$. We then have*

$$f(1) = \sum_{\mathfrak{F}\text{ forest}} \int_{[0,1]^{\mathfrak{F}}} d\vec{h}\ \frac{\partial^{|\mathfrak{F}|} f}{\prod_{l\in\mathfrak{F}} \partial s_l} \left(s(\mathfrak{F},\vec{h})\right)$$

where the sum is over all forests $\mathfrak{F}$ with vertex set E, the notation $d\vec{h}$ is for the Lebesgue measure on the set of parameters $[0,1]^{\mathfrak{F}}$, the partial derivatives of f are with respect to the entries indexed by the pairs belonging to $\mathfrak{F}$, and the evaluation of these derivatives is at the $\vec{h}$ dependent point $s(\mathfrak{F},\vec{h})$. Such points belong to $\mathcal{K}_E$.

Note that the empty forest always occurs and its contribution is $f(0) = f(v_{\hat{0}})$. There are many proofs of this result [8, 1, 9], but we believe the most natural and most easily generalizable is the one given in [2, §2]. It is recalled here for the sake of completeness. We will first prove an ordered forest analog of Theorem 1. A possibly empty sequence $F = (l_1, \ldots, l_p)$ of pairs l_i in $E^{(2)}$ is called an ordered forest or o-forest if the corresponding set $\mathfrak{F} = \{l_1, \ldots, l_p\}$ is a forest. Let us denote by Δ_p the simplex $\{\vec{\rho} \in \mathbb{R}^p | 1 \geq \rho_1 \geq \cdots \geq \rho_p \geq 0\}$. Given a vector of parameters $\vec{\rho} = (\rho_1, \ldots, \rho_p)$ in Δ_p we also define the multiplet $t(F,\vec{\rho}) = (t(F,\vec{\rho})_l)_{l\in E^{(2)}}$ as follows. Let $l = \{a,b\}$, with a and b distinct elements in E. If a and b fall in distinct connected components of the full forest $\{l_1, \ldots, l_p\}$ then we set $t(F,\vec{\rho})_l = 0$. Else we let $t(F,\vec{\rho})_l = \rho_q$ where q is the smallest index $1 \leq q \leq p$ such that a and b are connected by the subforest $\{l_1, l_2, \ldots, l_q\}$. The following important property is an easy consequence of this definition.

Proposition 1 *For any o-forest $F = (l_1, \dots, l_p)$ and vector $\vec{\rho} \in \Delta_p$, the multiplet $t(F, \vec{\rho})$ belongs to $\mathcal{K}_E = \bar{\Omega}_E$.*

Proof: Indeed one can write the convex combination

$$t(F, \vec{\rho}) = (1 - \rho_1) v_{\hat{0}} + (\rho_1 - \rho_2) v_{\pi_1} + \dots + (\rho_{r-1} - \rho_r) v_{\pi_{r-1}} + \rho_r v_{\pi_r}$$

where π_q is the partition of connected components of the forest $\{l_1, l_2, \dots, l_q\}$. Note that we used the fact that $\pi_0 = \hat{0}$. ■

Theorem 2 *Under the same hypotheses as in the previous theorem one has*

$$f(1) = \sum_{\substack{F=(l_1,\dots,l_p) \\ \text{o-forest}}} \int_{\Delta_p} d\vec{\rho} \, \frac{\partial^p f}{\partial s_{l_1} \dots \partial s_{l_p}} \left(t(F, \vec{\rho})\right)$$

where the summation allows all possible lenghts p for the o-forest F, including $p = 0$.

Proof: We will prove by induction on $r \geq 1$ that

$$f(1) = \sum_{p<r} \sum_{\substack{F=(l_1,\dots,l_p) \\ \text{o-forest}}} \int_{\Delta_p} d\vec{\rho} \, \frac{\partial^p f}{\partial s_{l_1} \dots \partial s_{l_p}} \left(t(F, \vec{\rho})\right)$$

$$+ \sum_{\substack{F=(l_1,\dots,l_r) \\ \text{o-forest}}} \int_{\Delta_r} d\vec{\rho} \, \frac{\partial^r f}{\partial s_{l_1} \dots \partial s_{l_r}} \left(\hat{t}(F, \vec{\rho})\right) \tag{1}$$

where $\hat{t}(F, \vec{\rho})_l$ is defined in the same way as $t(F, \vec{\rho})_l$ except that if l does not fall inside a connected component of $\{l_1, \dots, l_r\}$ one sets $\hat{t}(F, \vec{\rho})_l$ equal to the last parameter ρ_r instead of zero. Note that $\hat{t}(F, \vec{\rho})$ is still in the convex $\mathcal{K}_E$, since

$$\begin{aligned} \hat{t}(F, \vec{\rho}) &= t(F, \vec{\rho}) + \rho_r (v_{\hat{1}} - v_{\pi_r}) \\ &= (1 - \rho_1) v_{\hat{0}} + (\rho_1 - \rho_2) v_{\pi_1} + \dots + (\rho_{r-1} - \rho_r) v_{\pi_{r-1}} + \rho_r v_{\hat{1}} \ . \end{aligned} \tag{2}$$

Now by the Fundamental Theorem of Calculus,

$$f(1) = f(v_{\hat{1}}) = f(v_{\hat{0}}) + \int_0^1 d\rho_1 \, \frac{d}{d\rho_1} f(\rho_1 v_{\hat{1}} + (1 - \rho_1) v_{\hat{0}})$$

$$= f(0) + \sum_{l_1 \in E^{(2)}} \int_0^1 d\rho_1 \frac{\partial f}{\partial s_{l_1}} (\rho_1 v_{\hat{1}})$$

which is Eq. (1) for $r = 1$, as can be checked form the given definitions. Suppose (1) has been proven for $r \geq 1$, and consider the integrand

$$\frac{\partial^r f}{\partial s_{l_1} \dots \partial s_{l_r}} \left(\hat{t}(F, \vec{\rho})\right)$$

of any particular term in the second sum. Using (2) and introducing a new parameter ρ_{r+1}, we rewrite the argument of the derivative of f as

$$\hat{t}(F,\vec{\rho}) = (1-\rho_1)v_{\hat{0}} + (\rho_1-\rho_2)v_{\pi_1} + \cdots$$
$$+(\rho_{r-1}-\rho_r)v_{\pi_{r-1}} + (\rho_r-\rho_{r+1})v_{\pi_r} + \rho_{r+1}v_{\hat{1}}\ \Big|_{\rho_{r+1}=\rho_r}$$

We again use the Fundamental Theorem of Calculus with respect to ρ_{r+1} to interpolate between $\rho_{r+1}=\rho_r$ and $\rho_{r+1}=0$, hence

$$\frac{\partial^r f}{\partial s_{l_1}\ldots\partial s_{l_r}}\left(\hat{t}(F,\vec{\rho})\right) = \frac{\partial^r f}{\partial s_{l_1}\ldots\partial s_{l_r}}\left(t(F,\vec{\rho})\right)$$
$$+\int_0^{\rho_r} d\rho_{r+1}\sum_{l_{r+1}}\frac{\partial^{r+1} f}{\partial s_{l_1}\ldots\partial s_{l_r}\partial s_{l_{r+1}}}\left(\hat{t}((F,l_{r+1}),(\vec{\rho},\rho_{r+1}))\right)$$

where the sum is over pairs l_{r+1} corresponding to the nonzero entries of $v_{\hat{1}}-v_{\pi_r}$. This is tantamount to summing over all pairs not falling inside a connected component of F, i.e., all pairs one can append to F in order to produce an o-forest of length $r+1$. This immediately implies identity (1) for $r+1$. Finally, the identity (1) reduces to the statement of Theorem 2 as soon as r reaches $|E|-1$. This is because when $r=|E|-1$ the second sum in (1) is over ordered connecting trees F for which $\hat{t}(F,\vec{\rho})$ and $t(F,\vec{\rho})$ are the same. ■

Proof of Theorem 1: Starting from the identity in Theorem 2 we collect the o-forests corresponding to an unordered forest $\mathfrak{F}=\{l_1,\ldots,l_p\}$. This contributes

$$\sum_{\sigma\in\mathfrak{S}_p}\int_{1>\rho_1>\cdots>\rho_p>0} d\vec{\rho}\ \frac{\partial^{|\mathfrak{F}|} f}{\prod_{l\in\mathfrak{F}}\partial s_l}\left(t(F^\sigma,\vec{\rho})\right) \qquad (3)$$

where we ignored some set of measure zero, and used the notation $F^\sigma = (l_{\sigma(1)},\ldots,l_{\sigma(p)})$. Now define the family (rather than sequence) of variables $\vec{h}=(h_l)_{l\in\mathfrak{F}}$ by letting $h_{\sigma(q)}=\rho_q$ for any q, $1\le q\le p$. One can check from the previous definitions that

$$t(F^\sigma,\vec{\rho}) = s(\mathfrak{F},\vec{h})$$

As a result, the quantity (3) becomes

$$\sum_{\sigma\in\mathfrak{S}_p}\int_{1>h_{l_{\sigma(1)}}>\cdots>h_{l_{\sigma(p)}}>0} d\vec{h}\ \frac{\partial^{|\mathfrak{F}|} f}{\prod_{l\in\mathfrak{F}}\partial s_l}\left(s(\mathfrak{F},\vec{h})\right) = \int_{[0,1]^{\mathfrak{F}}} d\vec{h}\ \frac{\partial^{|\mathfrak{F}|} f}{\prod_{l\in\mathfrak{F}}\partial s_l}\left(s(\mathfrak{F},\vec{h})\right)$$

by combining the simplices of integration accounting for the relative ordering of the parameters into the full cube $[0,1]^{\mathfrak{F}}$. The statement about the arguments $s(\mathfrak{F},\vec{h})$ belonging to $\mathcal{K}_E$ follows from Prop. 1. ■

We will now deduce a few lemmas as corollaries of the BKAR forest formula.

Lemma 1 *Again let us consider a finite set E and let us denote by $E^{(2)}$ the set of unordered pairs $l=\{a,b\}$ in E. Let $V_{\{a,b\}}$ be a collection of complex numbers indexed by $E^{(2)}$. Then*

$$\sum_{\mathbf{g}\rightsquigarrow E}\prod_{l\in\mathbf{g}}\left(e^{-V_l}-1\right) = \sum_{\substack{\mathfrak{T}\rightsquigarrow E\\ \mathfrak{T}\ \text{tree}}}\int_{[0,1]^{\mathfrak{T}}} d\vec{h}\ \left\{\prod_{l\in\mathfrak{T}}(-V_l)\right\}\ e^{-\sum_{l\in E^{(2)}} s(\mathfrak{T},\vec{h})_l V_l}\ . \qquad (4)$$

Here g *is summed over all simple graphs (i.e. subsets of* $E^{(2)}$*) which connect* E*. We abreviate this property by the notation* $\mathsf{g} \rightsquigarrow E$*. On the right-hand side the sum is on spanning trees* $\mathfrak{T}$ *which connect* E*. The notation* $s(\mathfrak{T}, \vec{h})$ *is as in Theorem 1.*

Proof: This is a consequence of Theorem 1 and the uniqueness of the Möbius inverse. Given any nonempty subset $X \subset E$, let $\gamma_1(X)$ denote the expression analogous to the left-hand side of (4) for X instead of E. Namely one sums over graphs $\mathsf{g} \subset X^{(2)}$ connecting X, and the V_l are the ones coming from E by restriction:

$$\gamma_1(X) = \sum_{\mathsf{g} \rightsquigarrow X} \prod_{l \in \mathsf{g}} \left(e^{-V_l} - 1\right) \ .$$

Likewise let $\gamma_2(X)$ be the expression analogous to the right-hand side of (4) for X instead of E. For any partition $\pi \in \Pi_E$ we define

$$c_1(\pi) = \prod_{X \in \pi} \gamma_1(X)$$

and

$$c_2(\pi) = \prod_{X \in \pi} \gamma_2(X) \ .$$

Let us also define

$$d(\pi) = \prod_{l \in E^{(2)}} \left[e^{-\mathbb{1}\{\exists X \in \pi, l \subset X\} \ V_l} \right] \ .$$

Let us denote the natural order relation on the partition lattice Π_E by $\preceq$, i.e., one writes $\pi \preceq \pi'$ if partition π is a refinement of partition π'. We will first show that

$$d(\pi) = \sum_{\pi' \preceq \pi} c_1(\pi') \ .$$

Indeed, writing

$$e^{-\mathbb{1}\{\exists X \in \pi, l \subset X\} \ V_l} = 1 + \mathbb{1}\{\exists X \in \pi, l \subset X\} \ (e^{-V_l} - 1)$$

and expanding one has

$$d(\pi) = \sum_{\mathsf{g} \subset E^{(2)}} \mathbb{1}\left\{ \begin{array}{c} \forall l \in \mathsf{g} \\ \exists X \in \pi, l \subset X \end{array} \right\} \prod_{l \in \mathsf{g}} \left(e^{-V_l} - 1\right) \ .$$

Let us denote by $\pi(\mathsf{g}) \in \Pi_E$ the partition of connected components of a graph $\mathsf{g} \in E^{(2)}$. We then have by collecting the outcome of the expansion with respect to the connected components

$$\begin{aligned} d(\pi) &= \sum_{\pi' \in \Pi_E} \sum_{\substack{\mathsf{g} \subset E^{(2)} \\ \pi(\mathsf{g}) = \pi'}} \mathbb{1}\left\{ \begin{array}{c} \forall l \in \mathsf{g} \\ \exists X \in \pi, l \subset X \end{array} \right\} \prod_{l \in \mathsf{g}} \left(e^{-V_l} - 1\right) \\ &= \sum_{\pi' \preceq \pi} \prod_{X \in \pi'} \left\{ \sum_{\mathsf{g} \rightsquigarrow X} \prod_{l \in \mathsf{g}} \left(e^{-V_l} - 1\right) \right\} \end{aligned}$$

as wanted. Now we also have

$$d(\pi) = \prod_{X\in\pi} \prod_{l\in X^{(2)}} e^{-V_l} \ .$$

For any such X we consider the function

$$f_X(s) = \prod_{l\in X^{(2)}} e^{-s_l V_l}$$

for multiplets $s = (s_l)_{l\in X^{(2)}}$ and apply Theorem 1 to it, thus obtaining

$$\begin{aligned}
\prod_{l\in X^{(2)}} e^{-V_l} &= f_X(1) \\
&= \sum_{\mathfrak{F}_X \text{ forest on } X} \int_{[0,1]^{\mathfrak{F}_X}} d\vec{h}_X \ \frac{\partial^{|\mathfrak{F}_X|} f_X}{\prod_{l\in\mathfrak{F}_X} \partial s_l} \left(s(\mathfrak{F}_X, \vec{h}_X) \right) \\
&= \sum_{\mathfrak{F}_X \text{ forest on } X} \int_{[0,1]^{\mathfrak{F}_X}} d\vec{h}_X \ \left\{ \prod_{l\in\mathfrak{F}_X} (-V_l) \right\} e^{-\sum_{l\in X^{(2)}} s(\mathfrak{F}_X, \vec{h}_X)_l V_l} \ .
\end{aligned}$$

Now again collecting the terms componentwise, one can rewrite the latter expression as

$$\prod_{l\in X^{(2)}} e^{-V_l} =$$

$$\sum_{\pi_X\in\Pi_X} \prod_{Y\in\pi_X} \left[\sum_{\substack{\mathfrak{T}_Y \rightsquigarrow Y \\ \mathfrak{T}_Y \text{ tree}}} \int_{[0,1]^{\mathfrak{T}_Y}} d\vec{h}_Y \ \left\{ \prod_{l\in\mathfrak{T}_Y} (-V_l) \right\} e^{-\sum_{l\in Y^{(2)}} s(\mathfrak{T}_Y, \vec{h}_Y)_l V_l} \right] .$$

This is because of the definition of the $s(\mathfrak{F}, \vec{h})_l$ in the BKAR formula. These vanish for pairs which are not inside a connected component. Whereas for pairs l which are inside a connected component, the $s(\mathfrak{F}, \vec{h}_l)$ only depend on the edges of the forest $\mathfrak{F}$ which are in that component. Now

$$\prod_{l\in X^{(2)}} e^{-V_l} = \sum_{\pi_X\in\Pi_X} \prod_{Y\in\pi_X} \gamma_2(Y)$$

and as a result

$$\begin{aligned}
d(\pi) &= \prod_{X\in\pi} \left(\sum_{\pi_X\in\Pi_X} \prod_{Y\in\pi_X} \gamma_2(Y) \right) \\
&= \sum_{\pi'\preceq\pi} \prod_{Y\in\pi'} \gamma_2(Y)
\end{aligned}$$

where we collected the blocks of the partitions π_X, for X a block of π, into a single partion π' of E which is refinement of π. Hence

$$d(\pi) = \sum_{\pi'\preceq\pi} c_1(\pi')$$

Therefore both c_1 and c_2 are Möbius inverses of d on the partition lattice Π_E, and they must be equal. Specializing to $c_1(\hat{1}) = c_2(\hat{1})$ proves the lemma. ■

The following tree graph inequality, initially due to Brydges, Battle and Federbush (see [5, 4, 18]) is useful when performing Mayer expansions for a gas of particles with unbounded interaction potential energies.

Lemma 2 *Under the same hypotheses as in Lemma 1, let us assume that the numbers V_l satisfy, in addition, the following stability hypothesis: there are nonnegative numbers U_a, for $a \in E$, such that for any subset $S \subset E$ one has*

$$\left|\sum_{l\in S^{(2)}} V_l\right| \le \sum_{a\in S} U_a \ .$$

Then the following inequality holds

$$\left|\sum_{\mathsf{g}\rightsquigarrow E}\prod_{l\in\mathsf{g}}\left(e^{-V_l}-1\right)\right| \le e^{\sum_{a\in E}U_a} \sum_{\substack{\mathfrak{T}\rightsquigarrow E\\ \mathfrak{T}\ \text{tree}}}\prod_{l\in\mathfrak{T}}|V_l| \ .$$

Proof: This is an easy consequence of Lemma 1 and the fact that $s(\mathfrak{T},\vec{h})$ appearing in the BKAR formula is in $\mathcal{K}_E$, i.e., is a convex combination of partition vectors. Indeed, for any given $\mathfrak{T}$ and $\vec{h}$ as in (4), one can find nonnegative numbers $\lambda_1,\ldots,\lambda_p$, satisfying $\sum_{q=1}^{p}\lambda_q = 1$, as well as partitions $\pi_1,\ldots,\pi_p \in \Pi_E$ such that

$$s(\mathfrak{T},\vec{h}) = \sum_{q=1}^{p}\lambda_q v_{\pi_q} \ .$$

Therefore

$$\begin{aligned}
e^{-\sum_{l\in E^{(2)}} s(\mathfrak{T},\vec{h})_l V_l} &= \exp\left[-\sum_{q=1}^{p}\lambda_q\sum_{l\in E^{(2)}}(v_{\pi_q})_l V_l\right]\\
&= \exp\left[-\sum_{q=1}^{p}\lambda_q\sum_{X\in\pi_q}\sum_{l\in X^{(2)}} V_l\right] \ .
\end{aligned}$$

From which one derives

$$\begin{aligned}
\left|e^{-\sum_{l\in E^{(2)}} s(\mathfrak{T},\vec{h})_l V_l}\right| &\le \exp\left[\sum_{q=1}^{p}\lambda_q\sum_{X\in\pi_q}\left|\sum_{l\in X^{(2)}} V_l\right|\right]\\
&\le \exp\left[\sum_{q=1}^{p}\lambda_q\sum_{X\in\pi_q}\sum_{a\in X} U_a\right]\\
&\le \exp\left[\sum_{a\in E} U_a\right]
\end{aligned}$$

using the stability hypothesis. Now the desired inequality clearly follows from the formula (4). ■

Lemma 3 *Under the same hypotheses as in lemma 1 we have*

$$\sum_{\mathrm{g} \rightsquigarrow E} \prod_{l \in \mathrm{g}} (-V_l) = \sum_{\substack{\mathfrak{T} \rightsquigarrow E \\ \mathfrak{T} \text{ tree}}} \int_{[0,1]^{\mathfrak{T}}} d\vec{h} \prod_{l \in \mathfrak{T}} (-V_l) \prod_{l \in E^{(2)} \setminus \mathfrak{T}} (1 - s(\mathfrak{T}, \vec{h})_l V_l) \ .$$

Proof: The proof follows the same lines as that of Lemma 1. This time

$$d(\pi) = \prod_{l \in E^{(2)}} [\mathbb{1}\{\exists X \in \pi, l \subset X\} \ (1 - V_l)]$$

and the function to which one applies the BKAR formula is

$$f_X(s) = \prod_{l \in X^{(2)}} (1 - s_l V_l) \ .$$

The rest of the argument based on the uniquess of the Möbius inverse is the same. ■

An immediate corollary is the so-called Penrose-Rota inequality which bounds Mayer coefficients by a sum over spanning trees [14, 20].

Some history: The formula discovered by Brydges and Kennedy [8] is the one stated in Lemma 1. They used 1 minus the parameters, so the coefficients of the V's in the exponential involve a maximum over the connecting path instead of a minimum. Their proof uses an explicit tree-sum solution for a differential equation of Hamilton-Jacobi type (see [6] for a nice presentation). It is inspired by the Wilson-Polshinski continuous renormalization group differential equation [22, 16]. The reason why such solutions of nonlinear differential equations can be expressed as sums over trees is explained from a combinatorial point of view, e.g., in [13]. The fundamental calculus version of Theorem 1 first appeared in [1]. It was there called the Brydges-Kennedy taylor forest formula. D. Brydges calls it the Abdesselam-Rivasseau formula in his lecture at the conference "Combinatorial Identities and Their Applications in Statistical Mechanics", Cambridge, April 2008:

`http://www.newton.ac.uk/programmes/CSM/csmw03.html`

In [19] the name Brydges-Kennedy-Abdesselam-Rivasseau formula was first used. The proof given in [1] is purely algebraic and relies on a combinatorial partial fraction decomposition identity. Two other proofs of this partial fraction identity by A. Abdesselam and V. Lafforgue can be found at:

`http://people.virginia.edu/ aa4cr/forestpage.html`

The proof given here originates from ideas of H. Knörrer, J. Magnen and V. Rivasseau. It was considerably generalized in [2] to expansions which allow hypergraphs and p-PI connectivity. A similar generalization was independently discovered by G. Poirot [15].

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