RENORMALIZATION GROUP TRAJECTORIES BETWEEN TWO FIXED POINTS

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Main reference: A. A. CMP 07'
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Outline:

1. Global dynamics of Wilson’s RG
2. Rigorous results (selection)
3. The BMS model
4. Good infinite-volume coordinates
5. Idea of the proof
6. Functional analysis, norms
7. Perspectives
1. Global Dynamics of Wilson’s Renormalization Group:

\[
\phi^4 \text{ model: } \int F \, D\phi \cdot \cdot \cdot e^{-\int_{\mathbb{R}^d} \left[ \frac{1}{2} \left( \nabla \phi(x) \right)^2 + \mu \phi(x)^2 + g \phi(x)^4 \right] \, dx}
\]

\( F \) in infinite-dimensional space of functions \( \mathbb{R}^d \rightarrow \mathbb{R} \) with Lebesgue measure on \( F \).
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QFT functional integrals: a challenge for mathematicians
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e. g., the $\phi^4$ model

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\int \mathcal{F} D\phi \cdots e^{- \int_{\mathbb{R}^d} \left[ \frac{1}{2} (\nabla \phi)^2(x) + \mu \phi(x)^2 + g \phi(x)^4 \right] dx}
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$$

$\mathcal{F}$ infinite-dimensional space of functions $\mathbb{R}^d \to \mathbb{R}$
$D\phi$ Lebesgue measure on $\mathcal{F}$
Construction by scaling limit of lattice theories on $(a\mathbb{Z})^d \subset \mathbb{R}^d \iff$ cut-off $\frac{1}{a}$ on momenta in Fourier space

rescale to unit lattice

approximants to continuum theory $=$ points in the space of all possible unit cut-off theories
Construction by scaling limit of lattice theories on 
\((a\mathbb{Z})^d \subset \mathbb{R}^d \iff \text{cut-off } \frac{1}{a} \text{ on momenta in Fourier space}\)

- rescale to unit lattice
- approximants to continuum theory = points in the space of all possible unit cut-off theories
- \(\text{RG} = \text{dynamical system on this space}\)
Construction by scaling limit of lattice theories on $(a\mathbb{Z})^d \subset \mathbb{R}^d \iff$ cut-off $\frac{1}{a}$ on momenta in Fourier space

- rescale to unit lattice
- approximants to continuum theory $=$ points in the space of all possible unit cut-off theories
- RG $=$ dynamical system on this space

$\nu$ measure on random $\phi$ with $\hat{\phi}(p) = 0$ if $|p| > 1$

- introduce magnification ratio $L > 1$
- split $\phi = \zeta + \phi_{\text{low}}$

\[
\begin{align*}
\zeta & \iff L^{-1} < |p| \leq 1 \\
\phi_{\text{low}} & \iff |p| \leq L^{-1}
\end{align*}
\]
integrating over $\zeta \rightarrow$ marginal probability distribution on $\phi_{\text{low}}$

rescale $\psi(x) = L[\phi]\phi_{\text{low}}(Lx) \rightarrow$ measure $d\nu'$
- integrate over $\zeta \rightarrow$ marginal probability distribution on $\phi_{\text{low}}$
- rescale $\psi(x) = L[\phi]_{\text{low}}(Lx) \rightarrow$ measure $d\nu'$
- RG map: $d\nu \rightarrow d\nu'$
- Integrate over $\zeta \rightarrow$ marginal probability distribution on $\phi_{\text{low}}$
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Important features: fixed points, eigenvalues of linearized RG around them, local stable & unstable manifolds...
► integrate over $\zeta \rightarrow$ marginal probability distribution on $\phi_{\text{low}}$
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► RG map: $d\nu \rightarrow d\nu'$

Important features: fixed points, eigenvalues of linearized RG around them, local stable & unstable manifolds.

Local features
- integrate over $\zeta$ $\longrightarrow$ marginal probability distribution on $\phi_{\text{low}}$
- rescale $\psi(x) = L[\phi] \phi_{\text{low}}(Lx) \longrightarrow$ measure $d\nu'$
- RG map: $d\nu \longrightarrow d\nu'$

Important features: fixed points, eigenvalues of linearized RG around them, local stable & unstable manifolds...

Local features

Global features ? e.g. heteroclinic trajectories between fixed points
2. Rigorous Results (Selection):

Local:
- RG exponents for Gaussian fixed point, \( \exists \) nontrivial IR fp in \( 4 - \epsilon \) dimensions, local stable/unstable manifolds, for HM: Bleher-Sinai CMP 73', 75', Collet-Eckmann CMP 77', LNP 78', Gawȩdzki-Kupiainen CMP 83', JSP 84', Pereira JMP 93'
- HM at \( \epsilon = 1 \): Koch-Wittwer CMP 86'
- New fps at \( d = 2 + 2n - 1, n = 3, 4, ... \) in LPA: Felder CMP 95'
- Euclidean model in \( 4 - \epsilon \) dimensions, \( \exists \) nontrivial IR fp and local stable manifold: Brydges-Dimock-Hurd CMP 98'
- Same for nicer model: Brydges-Mitter-Scoppola CMP 03'

Global:
- Uniqueness of IR fp in LPA for \( 3 \leq d < 4 \): Lima CMP 87'
- Massless GN in \( 2 + \epsilon \) dim: Gawȩdski-Kupiainen NPB 85'
- BMS model, construction of discrete heteroclinic trajectories joining Gaussian UV fp to nontrivial IR fp: A. A. CMP 07'
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3. The BMS Model:

$\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$

$Z = \int \mathcal{D}\phi e^{-\frac{1}{2} \langle \phi, (\nabla)^3 + \epsilon \phi^4 \rangle_{L^2(\mathbb{R}^3)}}$

Gaussian measure - potential $V(\phi)$

$\int dx (\mu : \phi^2(x) : + g : \phi^4(x) :)$

$\sim (\nabla)^3 - 3 + \epsilon \phi^4 (x, y) \sim 1 |x - y|^2 \phi$

$[\phi] = 3 - \epsilon \phi^4$

$\sim \int_0^\infty dl l^2 [\phi] u(x - y l)$

$u$ finite range, smooth, and nonnegative in $x$ and $p$

$C(x - y) = \int_1^\infty dl l^2 [\phi] u(x - y l)$
3. The BMS Model:

Scalar field $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$

$$Z = \int D\phi \ e^{-\frac{1}{2} \langle \phi, (-\Delta)^{3+\epsilon} \phi \rangle_{L^2(\mathbb{R}^3)}} - \int dx (\mu : \phi^2(x) : + g : \phi^4(x) : )$$

potential $V(\phi)$

Gaussian measure
3. The BMS Model:

Scalar field $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$

$$Z = \int D\phi \ e^{-\frac{1}{2} \langle \phi, (\Delta + \epsilon) \phi \rangle_{L^2(\mathbb{R}^3)}} - \int dx (\mu \phi^2(x) + g \phi^4(x))$$

- **Gaussian measure**

- **potential** $V(\phi)$

- **propagator** $(-\Delta)^{-\frac{3+\epsilon}{4}} (x, y) \sim \frac{1}{|x-y|^{2[\phi]}}$

- $[\phi] = \frac{3-\epsilon}{4}$

- **propagator** $\sim \int_0^\infty \frac{dl}{l} l^{-2[\phi]} u\left(\frac{x-y}{l}\right)$

- **$u$** finite range, smooth, and nonnegative in $x$ and $p$

- **unit cut-off** $C(x-y) = \int_1^\infty \frac{dl}{l} l^{-2[\phi]} u\left(\frac{x-y}{l}\right)$
split $C(x - y) = \Gamma(x - y) + C_{L^{-1}}(x - y)$

with $C_{L^{-1}}(x - y) = L^{-2[\phi]} C(L^{-1}(x - y))$ and

$$\Gamma(x - y) = \int_{1}^{L} \frac{dl}{l} l^{-2[\phi]} u \left( \frac{x - y}{l} \right)$$

convolution $d\mu_C = d\mu_\Gamma \ast d\mu_{C_{L^{-1}}}$

$$Z = \int d\mu_C(\phi) \mathcal{Z}(\phi) = \int d\mu_{C_{L^{-1}}}(\psi)d\mu_\Gamma(\zeta) \mathcal{Z}(\psi + \zeta)$$

$$= \int d\mu_C(\phi) (\mathcal{R}\mathcal{Z})(\phi)$$

$$(\mathcal{R}\mathcal{Z})(\phi) = \int d\mu_\Gamma(\zeta) \mathcal{Z}(\phi_{L^{-1}} + \zeta) \text{ and } \phi_{L^{-1}}(x) = L^{-[\phi]} \phi(L^{-1}x)$$
split \( C(x - y) = \Gamma(x - y) + C_{L-1}(x - y) \)

with \( C_{L-1}(x - y) = L^{-2[\phi]} C(L^{-1}(x - y)) \) and

\[
\Gamma(x - y) = \int_1^L \frac{dl}{l} l^{-2[\phi]} u \left( \frac{x - y}{l} \right)
\]

convolution \( d\mu_C = d\mu_{\Gamma} * d\mu_{C_{L-1}} \)

\[
Z = \int d\mu_C(\phi) \, Z(\phi) = \int d\mu_{C_{L-1}}(\psi) d\mu_{\Gamma}(\zeta) \, Z(\psi + \zeta)
\]

\[
= \int d\mu_C(\phi) \, (RZ)(\phi)
\]

\((RZ)(\phi) = \int d\mu_{\Gamma}(\zeta) \, Z(\phi_{L-1} + \zeta) \) and \( \phi_{L-1}(x) = L^{-[\phi]} \phi(L^{-1}x) \)

RG map: \( Z \longrightarrow \mathcal{R}Z \)
4. Good Infinite-Volume Coordinates:

Brydges et al.

\[ \mathbb{Z}^3 \subset \mathbb{R}^3 \implies \text{cell decomposition} \]
In finite box $\Lambda$

$$Z(\Lambda, \phi) =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{X_1, \ldots, X_n \text{ disjoint in } \Lambda}} \exp \left[ - \int_{\Lambda \setminus (\bigcup X_i)} dx \{ g : \phi^4(x) : c + \mu : \phi^2(x) : c \} \right]$$

$$\times K(X_1, \phi|X_1) \cdots K(X_n, \phi|X_n)$$

- $Z \longleftrightarrow (g, \mu, K)$
- $K = (K(X, \cdot))_X$ polymer collection of local functionals
In finite box $\Lambda$

$$Z(\Lambda, \phi) =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\text{disjoint in } \Lambda} \exp \left[ - \int_{\Lambda \setminus (\cup X_i)} dx \{ g : \phi^4(x) : c + \mu : \phi^2(x) : c \} \right]$$

$$\times K(X_1, \phi|X_1) \cdots K(X_n, \phi|X_n)$$

$\text{Z} \longleftrightarrow (g, \mu, K)$

$K = (K(X, \cdot))_X$ polymer collection of local functionals

need to extract second order perturbation theory:

$$K(X, \phi) = g^2[\text{explicit complicated formula}] e^{-V(X,\phi)} + R(X, \phi)$$

$R$ of order $g^3$
In finite box $\Lambda$

$$Z(\Lambda, \phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{X_1, \ldots, X_n \text{ disjoint in } \Lambda} \exp \left[ - \int_{\Lambda \setminus (\cup X_i)} dx \{ g : \phi^4(x) : c + \mu : \phi^2(x) : c \} \right] \times K(X_1, \phi|X_1) \cdots K(X_n, \phi|X_n)$$

$\blacktriangleright$ $Z \longleftrightarrow (g, \mu, K)$

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$\blacktriangleright$ need to extract second order perturbation theory:

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$\blacktriangleright$ $R$ of order $g^3$

$\blacktriangleright$ RG map: $(g, \mu, R) \longrightarrow (g', \mu', R')$
Sketch of the RG phase portrait:
RG map in \((g, \mu, R)\) coordinates:

\[
\begin{align*}
g' &= L^\epsilon g - L^{2\epsilon} a(L, \epsilon) g^2 + \xi_g(g, \mu, R) \\
\mu' &= L^{\frac{3+\epsilon}{2}} \mu + \xi_\mu(g, \mu, R) \\
R' &= \mathcal{L}^{(g, \mu)}(R) + \xi_R(g, \mu, R)
\end{align*}
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where \(\mathcal{L}^{(g,\mu)}(\cdot)\) is a contraction
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\end{align*}
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where \(\mathcal{L}^{(g, \mu)}(\cdot)\) is a contraction.

Simplified RG map:

\[
\begin{align*}
g' &= L^\varepsilon g - L^{2\varepsilon} a(L, \varepsilon) g^2 \\
\mu' &= L^{3+\varepsilon} \mu \\
R' &= \mathcal{L}^{(g, \mu)}(R)
\end{align*}
\]
Approximate fixed point at $(\bar{g}_*, 0, 0)$ with

$$\bar{g}_* = \frac{L^\epsilon - 1}{L^{2\epsilon} a} \sim \epsilon$$
Fake trajectory \((\bar{g}_n)_{n \in \mathbb{Z}}\) obtained by simple 1d, yet nonlinear, iteration by \(f(x) = L^\epsilon x - L^{2\epsilon} ax^2\)
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Note that \(f'(0) = L^\epsilon > 1\) and \(f'(\bar{g}_*) = 2 - L^\epsilon < 1\)
True trajectories \((g_n, \mu_n, R_n)_{n \in \mathbb{Z}}\) constructed as perturbations of fake trajectories
Theorem
(A. A. CMP 07’)

In the regime where \( \epsilon > 0 \) is small enough, for any \( \omega_0 \in ]0, \frac{1}{2}[ \), there exists a complete trajectory \((g_n, \mu_n, R_n)_{n \in \mathbb{Z}}\) for the RG map such that \( \lim_{n \to -\infty} (g_n, \mu_n, R_n) = (0, 0, 0) \) the Gaussian ultraviolet fixed point, and \( \lim_{n \to +\infty} (g_n, \mu_n, R_n) = (g_*, \mu_*, R_*) \) the BMS nontrivial infrared fixed point, and determined by the ‘initial condition’ at unit scale

\[ g_0 = \omega_0 \bar{g}_* \]
5. Idea of the Proof:

\[(g_n, \mu_n, R_n) = (\bar{g}_n + \delta g_n, \mu_n, R_n)\]

\[\text{rewrite } RG \text{ in terms of deviation variables } (\delta g_n, \mu_n, R_n)\]

\[\delta g_{n+1} = f'(\bar{g}_n)\delta g_n + \left[-L_2 \epsilon a \delta g_n^2 + \xi g_n(\bar{g}_n + \delta g_n, \mu_n, R_n)\right], \mu_{n+1} = L_3 + \epsilon_2 \mu_n + \xi \mu_n(\bar{g}_n + \delta g_n, \mu_n, R_n), \]

\[R_{n+1} = L(\bar{g}_n + \delta g_n, \mu_n)\]

Boundary conditions:

\[\text{Infrared: } \mu_n \text{ does not blow up when } n \to +\infty\]

\[\text{Ultraviolet: } R_n \text{ does not blow up when } n \to -\infty\]

\[\text{Anthropic? } \delta g_0 = 0\]
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- rewrite RG in terms of deviation variables \((\delta g_n, \mu_n, R_n)\)

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\begin{align*}
\delta g_{n+1} &= f'(\bar{g}_n)\delta g_n + \left[-L^2 \epsilon a \delta g_n^2 + \xi_g(\bar{g}_n + \delta g_n, \mu_n, R_n)\right], \\
\mu_{n+1} &= L^{3+\epsilon} \mu_n + \xi_\mu(\bar{g}_n + \delta g_n, \mu_n, R_n), \\
R_{n+1} &= L'(\bar{g}_n + \delta g_n, \mu_n)(R_n) + \xi_R(\bar{g}_n + \delta g_n, \mu_n, R_n)
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5. Idea of the Proof:

- $(g_n, \mu_n, R_n) = (\bar{g}_n + \delta g_n, \mu_n, R_n)$
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\mu_{n+1} & = L \frac{3+\epsilon}{2} \mu_n + \xi \mu(\bar{g}_n + \delta g_n, \mu_n, R_n), \\
R_{n+1} & = \mathcal{L}(\bar{g}_n + \delta g_n, \mu_n)(R_n) + \xi_R(\bar{g}_n + \delta g_n, \mu_n, R_n)
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Boundary conditions:

- Infrared: $\mu_n$ does not blow up when $n \to +\infty$
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- rewrite RG in terms of deviation variables \((\delta g_n, \mu_n, R_n)\)

\[
\begin{align*}
\delta g_{n+1} &= f'(\bar{g}_n)\delta g_n + \left[ -L^2 a \delta g_n^2 + \xi_g(\bar{g}_n + \delta g_n, \mu_n, R_n) \right], \\
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Boundary conditions:

- Infrared: \(\mu_n\) does not blow up when \(n \to +\infty\)
- Ultraviolet: \(R_n\) does not blow up when \(n \to -\infty\)
- Anthropic?: \(\delta g_0 = 0\)
Forward and backward integral equations towards the boundary conditions

\(\forall n > 0,\)
\[\delta g_n = f'(\bar{g}_{n-1})\delta g_{n-1} + \left[-L^{2\epsilon} a \delta g_{n-1}^2 + \xi_g(\bar{g}_{n-1} + \delta g_{n-1}, \mu_{n-1}, R_{n-1})\right]\]

\(\forall n < 0,\)
\[\delta g_n = \frac{1}{f'(\bar{g}_n)} \delta g_{n+1} - \frac{1}{f'(\bar{g}_n)} \left[-L^{2\epsilon} a \delta g_n^2 + \xi_g(\bar{g}_n + \delta g_n, \mu_n, R_n)\right]\]

\(\forall n \in \mathbb{Z},\)
\[\mu_n = L^{-\left(\frac{3+\epsilon}{2}\right)} \mu_{n+1} - L^{-\left(\frac{3+\epsilon}{2}\right)} \xi_{\mu}(\bar{g}_n + \delta g_n, \mu_n, R_n)\]

\(\forall n \in \mathbb{Z},\)
\[R_n = L^{\left(\bar{g}_{n-1} + \delta g_{n-1}, \mu_{n-1}\right)}(R_{n-1}) + \xi_R(\bar{g}_{n-1} + \delta g_{n-1}, \mu_{n-1}, R_{n-1})\]
Forward and backward integral equations towards the boundary conditions

∀ \( n > 0 \),

\[
\delta g_n = f'(\bar{g}_{n-1})\delta g_{n-1} + \left[ -L^{2\epsilon} a \delta g_{n-1}^2 + \xi_g(\bar{g}_{n-1} + \delta g_{n-1}, \mu_{n-1}, R_{n-1}) \right]
\]

∀ \( n < 0 \),

\[
\delta g_n = \frac{1}{f'(\bar{g}_n)} \delta g_{n+1} - \frac{1}{f'(\bar{g}_n)} \left[ -L^{2\epsilon} a \delta g_n^2 + \xi_g(\bar{g}_n + \delta g_n, \mu_n, R_n) \right]
\]

∀ \( n \in \mathbb{Z} \),

\[
\mu_n = L^{-\left(\frac{3+\epsilon}{2}\right)} \mu_{n+1} - L^{-\left(\frac{3+\epsilon}{2}\right)} \xi_{\mu}(\bar{g}_n + \delta g_n, \mu_n, R_n)
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∀ \( n \in \mathbb{Z} \),

\[
R_n = L^{\left(\bar{g}_{n-1}+\delta g_{n-1}, \mu_{n-1}\right)}(R_{n-1}) + \xi_R(\bar{g}_{n-1} + \delta g_{n-1}, \mu_{n-1}, R_{n-1})
\]

then iterate until hit the b.c.
Fixed point equation in space of sequences:

∀ \( n > 0 \),

\[
\delta g_n = \sum_{0 \leq p < n} \left( \prod_{p < j < n} f'(\bar{g}_j) \right) \left[ -L^{2\epsilon} a \, \delta g_p^2 + \xi_g(\bar{g}_p + \delta g_p, \mu_p, R_p) \right]
\]

∀ \( n < 0 \),

\[
\delta g_n = -\sum_{n \leq p < 0} \left( \prod_{n \leq j \leq p} \frac{1}{f'(\bar{g}_j)} \right) \left[ -L^{2\epsilon} a \, \delta g_p^2 + \xi_g(\bar{g}_p + \delta g_p, \mu_p, R_p) \right]
\]

∀ \( n \in \mathbb{Z} \),

\[
\mu_n = -\sum_{p \geq n} L^{-\left(\frac{3+\epsilon}{2}\right)(p-n+1)} \xi_{\mu}(\bar{g}_p + \delta g_p, \mu_p, R_p)
\]

∀ \( n \in \mathbb{Z} \),

\[
R_n = \sum_{p < n} \mathcal{L}(\bar{g}_{n-1} + \delta g_{n-1}, \mu_{n-1}) \circ \mathcal{L}(\bar{g}_{n-2} + \delta g_{n-2}, \mu_{n-2}) \circ \ldots
\]

\[
\ldots \circ \mathcal{L}(\bar{g}_{p+1} + \delta g_{p+1}, \mu_{p+1}) (\xi_R(\bar{g}_p + \delta g_p, \mu_p, R_p))
\]
Use this to define a map on a space of sequences \((\delta g_n, \mu_n, R_n)_{n \in \mathbb{Z}}\)
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\(\forall n > 0,\)

\[
\delta g'_n = \sum_{0 \leq p < n} \left( \prod_{p < j < n} f'(\bar{g}_j) \right) \left[ -L^{2\epsilon} a \, \delta g_p^2 + \xi g(\bar{g}_p + \delta g_p, \mu_p, R_p) \right]
\]

\(\forall n < 0,\)

\[
\delta g'_n = -\sum_{n \leq p < 0} \left( \prod_{n \leq j \leq p} \frac{1}{f'(\bar{g}_j)} \right) \left[ -L^{2\epsilon} a \, \delta g_p^2 + \xi g(\bar{g}_p + \delta g_p, \mu_p, R_p) \right]
\]

\(\forall n \in \mathbb{Z},\)

\[
\mu'_n = -\sum_{p \geq n} L^{-\left(\frac{3+\epsilon}{2}\right)(p-n+1)} \xi \mu(\bar{g}_p + \delta g_p, \mu_p, R_p)
\]

\(\forall n \in \mathbb{Z},\)

\[
R'_n = \sum_{p < n} \mathcal{L}(\bar{g}_{n-1} + \delta g_{n-1}, \mu_{n-1}) \circ \mathcal{L}(\bar{g}_{n-2} + \delta g_{n-2}, \mu_{n-2}) \circ \ldots
\]

\[
\ldots \circ \mathcal{L}(\bar{g}_{p+1} + \delta g_{p+1}, \mu_{p+1}) \left( \xi_R(\bar{g}_p + \delta g_p, \mu_p, R_p) \right)
\]
Use contraction mapping argument in a big Banach space of double-sided sequences
Use contraction mapping argument in a big Banach space of double-sided sequences

The fixed point (in the space of sequences) is a trajectory
6. Functional Analysis, Norms:

Fields:

- ▶ $\Delta$ a closed cube. Sobolev imbedding $W^{4,2}(\Delta) \hookrightarrow C^2(\Delta)$
- ▶ $\phi \in \text{Fld}(X) = \bigoplus_{\Delta \subset X} W^{4,2}(\hat{\Delta})$ plus $C^2$ gluing conditions
- ▶ $\|\phi\|_{\text{Fld}(X)} = \left(\sum_{\Delta \subset X} \sum_{|\nu| \leq 4} \|\partial^{\nu} \phi_{\Delta}\|_{L^2(\hat{\Delta})}^2\right)^{\frac{1}{2}}$
- ▶ Fluctuation measure $d\mu_\Gamma$ realized in Hilbert spaces $\text{Fld}(X)$
- ▶ Also need $\|\phi\|_{C^2(X)} = \sup_{x \in X} \max_{|\nu| \leq 2} |\partial^{\nu} \phi(x)|$
- ▶ $\phi$’s are real-valued
Functionals:
Functionals:

$$
\|K\| = \sup_{\Delta_0} \sum_{X \supset \Delta_0} L^5|X| \sup_{\phi \in \text{Fld}(X)} \left\{ e^{-\kappa \sum_{\Delta \subset X} \sum_{1 \leq |\nu| \leq 4} \|\partial^\nu \phi\|^2_{L^2(\Delta)}} \right\}
$$

$$
\times \sum_{0 \leq n \leq 9} \frac{(cg^{-\frac{1}{4}})^n}{n!} \sup_{\phi_1, \ldots, \phi_n \in \text{Fld}(X) \setminus \{0\}} \left| D^n K(X, \phi; \phi_1, \ldots, \phi_n) \right| \left\{ \|\phi_1\|_{C^2(\Delta)} \cdots \|\phi_n\|_{C^2(\Delta)} \right\}
$$

$K$'s allowed to be complex-valued

Fibered norm problem: norms depend on the dynamical variable $g$

use fake solution to calibrate
Functionals:

$$\| K \| = \sup_{\Delta_0} \sum_{X \supset \Delta_0} L^5|X| \sup_{\phi \in \text{Fld}(X)} \left\{ \exp \left( \sum_{\Delta \subset X} \sum_{1 \leq |\nu| \leq 4} \| \partial^{\nu} \phi \|_{L^2(\Delta)}^2 \right) \right\}$$

$$\times \sum_{0 \leq n \leq 9} \frac{(cg^{-\frac{1}{4}})^n}{n!} \sup_{\phi_1, \ldots, \phi_n \in \text{Fld}(X) \backslash \{0\}} \frac{|D^n K(X, \phi; \phi_1, \ldots, \phi_n)|}{\| \phi_1 \|_{C^2(\Delta)} \cdots \| \phi_n \|_{C^2(\Delta)}}$$

- $K$’s allowed to be complex-valued
Functionals:

\[
\|K\| = \sup_{\Delta_0} \sum_{X \subseteq \Delta_0} L^5 |X| \sup_{\phi \in \text{Fld}(X)} \left\{ e^{-\kappa \sum_{\Delta \subseteq X} \sum_{1 \leq |\nu| \leq 4} \|\partial^\nu \phi\|^2_{L^2(\Delta)}} \right\}
\]

\[
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\]

- \( K \)'s allowed to be complex-valued
- **Fibered norm problem**: norms depend on the dynamical variable \( g \)
Functionals:

\[
\|K\| = \sup_{\Delta_0} \sum_{X \subseteq \Delta_0} L^5|X| \sup_{\phi \in \text{Fld}(X)} \left\{ e^{-\kappa \sum_{\Delta \subseteq X} \sum_{1 \leq |\nu| \leq 4} \|\partial^\nu \phi\|^2_{L^2(\Delta)}} \right\}
\]

\[
\times \sum_{0 \leq n \leq 9} \left( cg^{-\frac{1}{4}} \right)^n \frac{n!}{n!} \sup_{\phi_1, \ldots, \phi_n \in \text{Fld}(X) \setminus \{0\}} \frac{|D^n K(X, \phi; \phi_1, \ldots, \phi_n)|}{\|\phi_1\|_{C^2(\Delta)} \cdots \|\phi_n\|_{C^2(\Delta)}} \right\}
\]

- \(K\)'s allowed to be complex-valued
- Fibered norm problem: norms depend on the dynamical variable \(g\)
- use fake solution to calibrate
7. Perspectives:

▶ Study more refined dynamical systems features: construct full invariant curve, regularity properties, smoothness at $g_0$ (asked by K. Gawędzki), first on HM with Ph. D. student Ajay Chandra

▶ Correlation functions

▶ Composite fields, anomalous dimensions, preliminary results by P. K. Mitter

▶ Analytic continuation to Minkowski

▶ Alternate construction using phase space expansion of A. A. Ph. D. Thesis ’97’

▶ $2d$ $\sigma$-model

▶ GN in $2d$

▶ $\phi^4_3$

▶ $\phi^4_4$ (noncommutative)

▶ $2d$-model

▶ YM in $4d$

▶ Nontrivial UV fp in quantum gravity. . .
7. Perspectives:

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