1 Recursive Formulations

We wish to solve the following problem:

$$\max_{\{a_t, x_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t r(x_t, a_t)$$  \hspace{1cm} (1)

subject to

$$a_t \in \Gamma(x_t)$$  \hspace{1cm} (2)

$$x_{t+1} = t(x_t)$$  \hspace{1cm} (3)

$$x_0 \text{ given.}$$  \hspace{1cm} (4)

The horizon length, $T$, can be infinite (in fact, it is exactly when it is infinite that this approach is very useful, but the finite horizon case gives us useful insight). The reason we need a new method to solve this problem – rather than simply taking Kuhn-Tucker conditions – is that such a problem will have a huge number of decision variables if the horizon is large. If it is in fact infinite we cannot write down enough equations to completely solve our model. We must attack this problem in a new manner – the method of dynamic programming.

This approach rewrites the problem into one which has only one choice rather than an infinite number; we begin by discussing the components of a dynamic program and how to set them up.

A deterministic stationary discounted dynamic programming problem consists of five basic objects. These objects are

- The **state space** $X$ (a state $x$ is a set of variables sufficient to tell a household everything needed to make decisions, such as prices and income);

- The **feasible control space** $A = \cup_{x \in X} \Gamma(x)$ and the **feasible control correspondence** $\Gamma : X \to 2^A$ (these are the choices $a$ that are permissible at a given point in time when state $x$ is realized – a budget set is the simplest example of a feasible control correspondence);

- The **return function** $r : X \times A \to \mathcal{R}$ (this is the reward in current terms for taking actions $a$ in state $x$);

- The **transition function** $t : X \times A \to X$ (this determines tomorrow’s state $x' = t(x, a)$ when action $a$ is taken in current state $x$);

- The **discount factor** $\beta$.

At this point, an example is instructive – we will choose the deterministic optimal growth model of Cass-Koopmans, which extended the famous Solow model to permit elastic savings rates. We will focus here on planning problems – later we will investigate competitive equilibria.
In this model, output is produced using capital only – the production technology is given by \( f(k_t) \). The representative household or planner chooses sequences of consumption \( \{c_t\}_{t=0}^T \) and capital \( \{k_{t+1}\}_{t=0}^T \) to maximize lifetime utility

\[
\sum_{t=0}^T \beta^t u(c_t)
\]

subject to the budget constraints

\[
c_t + k_{t+1} \leq f(k_t)
\]

and a given initial capital stock \( k_0 \). As a simple exercise, we will formulate all five components of our dynamic programming problem.

1. The state space: Current capital \( k_t \) is the only thing the planner needs to know – thus, the state vector \((k_t) \in K = \mathbb{R}_+ \) (capital must be nonnegative);

2. The feasible control space: The controls here are \((c_t, k_{t+1})\); they must lie in the set

\[
B(k_t) = \{(c_t, k_{t+1}) | c_t + k_{t+1} \leq f(k_t)\}
\]

that is, we cannot use on consumption plus savings more than we create in production; the feasible control correspondence is simply \( B(k_t) \);

3. The return function: this is simply the period utility function \( u(c_t) \);

4. The transition function: tomorrow’s state is the control \( k_{t+1} \), so the transition function is trivial;

5. The discount factor: simply \( \beta \).

So far we haven’t done anything particularly useful. However, I will show that this approach has value by solving a series of problems with progressively longer horizons, showing that the solutions display simple patterns. Using these patterns we will rewrite our problem recursively; that is, we will write it in a way that only depends on the current state and only has a choice for the current control.

Now we will proceed forward by solving this problem for \( T = 0 \) (a static problem); notation will be a bit of a pain, so let’s agree now that \( v^T_t(k_t) \) is the value function of a problem with horizon \( T \) in period \( t \) (when \( T = \infty \) we will simply drop the indices, because they won’t matter). Setting lifetime utility to zero after death (we are free to normalize death utility in any way we see fit) this problem becomes

\[
v^0_0(k_0) = \max_{k_1} \{u(f(k_0) - k_1)\}.
\]

We will impose the condition that \( k_1 \geq 0 \); that is, capital cannot be negative in the final period. This restriction is necessary for there to exist a solution if \( u \) is increasing, as
otherwise we would choose \( k_1 \) that is arbitrarily negative. Furthermore, we will specialize to the following functional forms because it makes the algebraic solution possible:

\[
f(k) = Ak^\alpha
\]

\[
u(c) = \log(c),
\]

and make the assumptions that \( \alpha \in [0, 1] \) and \( \beta \in (0, 1) \). The solution to the above problem is obviously

\[
k_1 = 0 \tag{11}
\]

\[
c_0 = Ak_0^\alpha. \tag{12}
\]

The solution is trivial; the planner tells the household to eat everything and then go off to die. The value function is therefore

\[
v_0^0(k_0) = \log(Ak_0^\alpha) = \log(A) + \alpha \log(k_0). \tag{13}
\]

Lifetime utility depends on the existing stock of capital; endowing an economy with more capital will generate more utility for consumers in this static world.

Now, examine the problem for \( T = 1 \). This problem is

\[
v_1^0(k_0) = \max_{k_1, k_2} \{ \log(Ak_0^\alpha - k_1) + \beta \log(Ak_1^\alpha - k_2) \} \tag{14}
\]

subject to nonnegativity on \( k_2 \) (but not necessarily on \( k_1 \)). Clearly we wish to set \( k_2 = 0 \). The first-order condition for \( k_1 \) is

\[
\frac{1}{Ak_0^\alpha - k_1} = \frac{\beta A\alpha k_1^{\alpha-1}}{Ak_1^\alpha} \Rightarrow k_1 = \frac{\alpha \beta}{1 + \alpha \beta} Ak_0^\alpha. \tag{15}
\]

The value of making these decisions is

\[
v_1^0(k_0) = \log \left( \frac{Ak_0^\alpha - \alpha \beta}{1 + \alpha \beta} Ak_0^\alpha \right) + \beta \log \left( A \left( \frac{\alpha \beta}{1 + \alpha \beta} Ak_0^\alpha \right)^\alpha \right)
\]

\[
= \log \left( \frac{1}{1 + \alpha \beta} A \right) + \alpha \log(k_0) + \beta \log \left( A \left( \frac{\alpha \beta}{1 + \alpha \beta} A \right)^\alpha \right) + \alpha^2 \beta \log(k_0).
\]

Once again the value is increasing in the initial capital stock \( k_0 \); if we give people more capital they will be better off. Notice that

\[
v_0^1(k_0) = \log(A) + \alpha \log(k_0) + \log \left( \frac{1}{1 + \alpha \beta} \right) + \beta \log \left( A \left( \frac{\alpha \beta}{1 + \alpha \beta} A \right)^\alpha \right) + \alpha^2 \beta \log(k_0)
\]

\[
= v_1^0(k_0) + \log \left( \frac{1}{1 + \alpha \beta} \right) + \alpha \beta \log \left( \frac{\alpha \beta}{1 + \alpha \beta} \right) + (1 + \alpha) \beta \log(A) + \alpha^2 \beta \log(k_0);
\]

that is, the value function for the two-period case is the value function for the static case plus some extra terms. It is easy to see that

\[
v_1^1(k_0) = \max_{k_1} \{ \log(Ak_0^\alpha - k_1) + \beta \log(A) + \alpha \log(k_1) \}
\]
so that we can write it recursively as
\[
v_0^1(k_0) = \max_{k_1} \left\{ \log (A k_0^\alpha - k_1) + \beta v_0^1(k_1) \right\}.
\]

Finally, let’s examine the case \( T = 2 \); this problem is given by
\[
v_0^2(k_0) = \max_{k_1, k_2, k_3} \left\{ \log (A k_0^\alpha - k_1) + \beta \log (A k_1^\alpha - k_2) + \beta^2 \log (A k_2^\alpha - k_3) \right\}
\]
subject to nonnegativity on \( k_3 \). Once again, we set \( k_3 = 0 \). The first-order conditions for \( k_1 \) and \( k_2 \) are given by
\[
\begin{align*}
\frac{1}{Ak_0^\alpha - k_1} &= \frac{\alpha \beta k_1^{\alpha-1}}{Ak_1^\alpha - k_2}, \\
\beta &= \frac{\alpha \beta^2 k_2^{\alpha-1}}{Ak_2^\alpha}. 
\end{align*}
\]
The solution to these equations is
\[
\begin{align*}
k_1 &= \frac{\alpha \beta + (\alpha \beta)^2}{1 + \alpha \beta + (\alpha \beta)^2} Ak_0^\alpha, \\
k_2 &= \frac{\alpha \beta}{1 + \alpha \beta} Ak_1^\alpha.
\end{align*}
\]
The value function for this problem is a big mess
\[
v_0^2(k_0) = \log \left( \frac{1}{1 + \alpha \beta + (\alpha \beta)^2} Ak_0^\alpha \right) + \beta \log \left( \frac{1}{1 + \alpha \beta} \right) + \beta^2 \log \left( \frac{\alpha \beta}{1 + \alpha \beta + (\alpha \beta)^2} \right) + A^{1+\alpha} k_0^{\alpha^3} \]
and it again satisfies a recursive equation:
\[
v_0^2(k_0) = \max_{k_1} \left\{ \log (A k_0^\alpha - k_1) + \beta v_0^1(k_1) \right\}.
\]

The general solution to a problem with horizon \( T \) is
\[
\begin{align*}
k_1 &= \frac{\alpha \beta + \cdots (\alpha \beta)^T}{1 + \alpha \beta + \cdots (\alpha \beta)^T} Ak_0^\alpha, \\
&= \alpha \beta \left[ \frac{1 - (\alpha \beta)^T}{1 - (\alpha \beta)^T + 1} \right] Ak_0^\alpha.
\end{align*}
\]
Letting $T \to \infty$ we have the decision rule

$$k_1 = \alpha \beta Ak_0^\alpha.$$ 

Note: here we are using the partial sum formula

$$\sum_{t=0}^{T-1} p^t = \frac{1 - p^T}{1 - p}$$

if $|p| < 1$, which works given that $\alpha \in [0, 1]$ and $\beta \in (0, 1)$.

Some things are important to note here. One, examining the decision rules from the problems with horizons of 2 and 3 periods, we see that the only thing that matters is the current capital stock $k_t$. Another important observation is that the decision rules depend not on the current period $t$ but on the number of periods before the end $T - t$; that is, a household makes the same decisions $n$ periods from death no matter how long they have been alive, conditional on current capital. We have then what is known as a recursive problem; the state of the world is given by $k_t$ and it is sufficient to determine current behavior. Furthermore, we may drop the time subscripts in the infinite horizon case, as the consumer will always be infinitely far from death:

$$k_{t+1} = \alpha \beta Ak_t^\alpha$$

or the commonly-used notation

$$k' = \alpha \beta Ak^\alpha$$

where primes denote next period values.

What about the values of these problems? It turns out that the value function (function of existing capital) also converges in the infinite horizon case. Although messy, you can show that it converges to a function of the form

$$v(k) = a + b \log(k)$$

where

$$a = \frac{1}{1 - \beta} \left[ \log(A(1 - \alpha \beta)) + \frac{\alpha \beta}{1 - \alpha \beta} \log(A \alpha \beta) \right]$$

and

$$b = \frac{\alpha}{1 - \alpha \beta}.$$ 

Note that we have dropped the subscripts on the value function as well; since consumers decisions only depend on the current capital stock and the time until death, only the current capital stock matters.

Solving the problem this way is not very fast when we know the form of the value function. Note that the above value functions imply

$$v_0^1(k) = \max_{k_1} \{ \log(c_0) + \beta v_1^0(k_1) \}.$$
What this says is that we can solve our problem turning our utility into the sum of two parts – what we get today and what we get in the future, assuming we make the proper choices tomorrow; we then only need to worry about making the proper choice today. With an infinite horizon we have

\[ v(k_t) = \max_{k_{t+1}} \{ \log(Ak_t^\alpha - k_{t+1}) + \beta v(k_{t+1}) \}. \]

\(v(k_t)\) is the lifetime utility from having \(k_t\) units of capital. This equation – the celebrated **Bellman equation** named after inventor/discoverer Richard Bellman – gives us a convenient method for solving the problem: if we could somehow know the form of the value function (and that no other function satisfied this equation) we could simply insert it into the above problem, maximize, and be done. If this sounds too good to be true, well it almost is; knowing the form of the value function is generally impossible. For the above case, we could insert a guess of the form

\[ v(k) = a + b \log(k) \]

into the Bellman equation and take derivatives:

\[ \frac{1}{Ak_t^\alpha - k_{t+1}} = \frac{\beta b}{k_{t+1}}. \]

The solution to this is

\[ k_{t+1} = \frac{\beta b}{1 + \beta b} Ak_t^\alpha. \]

The only problem is that we don’t know \(b\). But if we insert our solution into the Bellman equation we get

\[ a + b \log(k_t) = \log \left( \frac{1}{1 + \beta b} Ak_t^\alpha \right) + \beta a + \beta b \log \left( \frac{\beta b}{1 + \beta b} Ak_t^\alpha \right) \]

where the max disappears because it is embedded within the choice for \(k_{t+1}\). The important thing here is to note that this must hold for every value of \(k_t\); this implies that all the coefficients for the constant terms and \(\log(k_t)\) must be the same on each side of the equation. For \(\log(k_t)\) this requires

\[ b = \alpha + \beta b \alpha \]

or

\[ b = \frac{\alpha}{1 - \alpha \beta} \]

which is exactly what we got before. Matching up the constants gives us

\[ a = \log \left( \frac{A}{1 + \beta b} \right) + \beta a + \beta b \log \left( \frac{\beta b A}{1 + \beta b} \right) \]

or

\[ a = \frac{1}{1 - \beta} \left[ \log \left( \frac{A}{1 + \beta b} \right) + \beta b \log \left( \frac{\beta b A}{1 + \beta b} \right) \right]. \]
Inserting our solution for \( b \) yields

\[
\begin{align*}
    a &= \frac{1}{1 - \beta} \left[ \log \left( \frac{A}{1 + \beta b} \right) + \beta b \log \left( \frac{\beta b A}{1 + \beta b} \right) \right] \\
    &= \frac{1}{1 - \beta} \left[ \log \left( \frac{A}{1 + \beta \alpha} \right) + \beta \frac{\alpha}{1 - \alpha \beta} \log \left( \frac{\beta \alpha}{1 - \alpha \beta} \right) \right] \\
    &= \frac{1}{1 - \beta} \left[ \log((1 - \alpha \beta)A) + \frac{\alpha \beta}{1 - \alpha \beta} \log (\alpha \beta A) \right].
\end{align*}
\]

We have completely solved the consumer’s problem now; with the given solution for \( b \) optimal capital accumulation is given by

\[
k_{t+1} = \alpha \beta k_t^\alpha,
\]

exactly what we got before. The only problem with this method is that there are a very small number of economic problems where we know the form of the value function; of these, some are impossible to solve for the coefficients analytically and others are simply not very interesting from an economic standpoint as they involve odd choices for parameters.

What if we don’t know the form of the value function (either because we don’t know it or it simply does not exist in closed-form)? Go back to the general Bellman equation

\[
v_t(k_t) = \max_{k_{t+1}} \{ u(f(k_t) - k_{t+1}) + \beta v_{t+1}(k_{t+1}) \}.
\]

In the above problem we know that the value function is constant over time but we don’t know its form. Can we still use this equation? In fact, we can and will. The key is the recursive nature of the value function. Imagine if we guessed that the value in period \( T + 1 \) was zero, as we did solving the problem the first way. Then the Bellman equation would imply that

\[
v_T(k_T) = u(f(k_T)).
\]

But now we know by Bellman recursion that

\[
v_{T-1}(k_{T-1}) = \max_{k_T} \{ u(f(k_{T-1}) - k_T) + \beta v_T(k_T) \}.
\]

That is, we update our guess \( v_T \) by replacing it with \( v_{T-1} \) after solving for \( k_T \) as a function of \( k_{T-1} \). That is,

\[
v_{T-1}(k_{T-1}) = u \left( f \left( k_{T-1} \right) - k_T \left( k_{T-1} \right) \right) + \beta v_T \left( k_T \left( k_{T-1} \right) \right)
\]

Then, according to the Bellman equation we must now have that

\[
v_{T-2}(k_{T-2}) = \max_{k_{T-1}} \{ u(f(k_{T-2}) - k_{T-1}) + \beta v_{T-1}(k_{T-1}) \}
\]

and so on. If we let \( T \to \infty \) we would hope to eventually get close to the value function of the infinite horizon problem

\[
v(k) = \max_{k'} \{ u(f(k) - k') + \beta v(k') \}.
\]
under certain conditions. What we will attempt to do in the next sections is prove that this algorithm will converge to the true value function, that this value function exists, and that we can say something about how it "looks." In effect, we will be searching for the "certain conditions" that make this the appropriate way to proceed.

2 Dynamic Programming

To prove the algorithm from the previous section works, we must employ some mathematics from the realm of functional analysis, the study of spaces comprised not of numbers but of functions. Take the Bellman equation

$$v(x) = \max_a \{ r(x,a) + \beta v(x') : a \in \Gamma(x), x' = t(x,a) \}$$

where I have returned to the general notation of a dynamic programming problem. The function $v(x)$ is the unknown of this equation. We do know that inserting a function into the right-hand-side for $v(\cdot)$ and performing the maximization gives us a new function for the left-hand-side; these functions are not necessarily the same. Let our generic guess be given by $w$ so as not to confuse it with the true value function $v$, which may not exist and which we certainly do not know. We can view the Bellman equation as mapping functions into functions, a functional operator. Calling this thing $T$, we have the operator $T$ takes a function $w : X \to \mathbb{R}$ and "turns it into" a function $Tw : X \to \mathbb{R}$ via the process

$$(Tw)(x) = \max_a \{ r(x,a) + \beta w(x') : a \in \Gamma(x), x' = t(x,a) \}.$$ 

Lest this all seem very strange, I can point out that differentiation and integration are also functional operators – they "transform" one function into another. For example, let $D$ be the differentiation operator. Then $Dx^2 = 2x$, a new function which is related to the old one via the operator $D$. Similarly, integration maps a function into a family of functions that differ by a constant (it is a functional correspondence), so that

$$I(2x) = \int 2xdx = x^2 + C.$$ 

The Bellman operator works exactly the same way – it transforms a function $w(x)$ into a new function $(Tw)(x)$. We know that, if the true value function exists, it satisfies the Bellman equation. That is, if we feed the Bellman operator $v$ we get $v$ back; $(Tv)(x) = v(x)$. In other words, the value function is a fixed point of the $T$ operator in the space of functions.

For example, one fixed point for the differentiation operator is the zero function:

$$D0 = 0.$$ 

Another one is $e^x$:

$$De^x = e^x.$$ 

8
If we could somehow prove that the Bellman operator had a fixed point, we would know that
the value function exists. If we could further prove that it only has one fixed point, then we
would know that the value function is that fixed point; otherwise, if there were more than
one we would need to find the best fixed point (meaning that we would need a way to rank
them). That is our first task.

Finally, one point of terminology. Assuming that we have enough structure that solutions
to the Bellman equation exist, there will be (at least) one action for each value of the state
that is optimal. Denote this mapping the **policy function** (or correspondence):

\[ a^* = \pi(x) \]

Policy functions tell you what to do in the event that the state is \( x \); note that, in the infinite
horizon case, policy functions are invariant (they are used in every period).

Now remember back to the Contraction Mapping Theorem – a strict contraction on a
complete normed vector space has a unique fixed point. This condition is perfect for us,
since it gives both existence and uniqueness as well as a constructive procedure for locating
the fixed point. Therefore, our task is to find conditions under which the Bellman operator
is a strict contraction on a complete normed vector space. We then have achieved a large
part of our goal; we will have proven that the value function exists, is the unique solution to
the Bellman equation, and that iterating on the Bellman operator will converge to the value
function from any initial guess. All that would be left would be to prove that any solution
to the Bellman equation solves the sequential version of the household problem. For some
problems, we will not be able to prove that the Bellman operator is a contraction; in that
case, if we can prove that it is monotone we will know that it possesses at least one fixed
point. For this class of problems, we know that iterating from something strictly higher
than the value function will converge to the best fixed point; in some cases one can prove
that iterating from below converges to the same fixed point, ruling out any others.

We now examine the conditions under which the Bellman operator will be a contraction.
Suppose \( \phi \) is a function in the space of continuous functions \( \Xi \) that satisfies
\( \phi(x) > 0 \ \forall x \in X \).

Let’s call the following the ”\( \phi \)-norm” of a function:

\[ \|f\|_\phi = \sup_x \left\{ \frac{|f(x)|}{\phi(x)} : x \in X \right\}. \]

That is, deflate the value of \( f \) by a particular positive function \( \phi \); formally, \( \|\cdot\|_\phi \) is a member
of the class of **weighted norms**. Denote by \( \Xi_\phi \) the subset of \( \Xi \) for which this norm is
finite. This subset equipped with the \( \phi \)-norm \( (\Xi_\phi, \|\cdot\|_\phi) \) is a complete metric space (the
proof is essentially the same as the proof that the space of bounded continuous functions
equipped with the sup-norm is complete, which can be found in the mathematical analysis
section of the notes) and is also a vector space (adding two members of \( \Xi_\phi \) together clearly
give you another member, and so does multiplying by a finite scalar), so we have a complete
normed vector space. We now have the following result, which extends the classic Blackwell
theorem to \( \Xi_\phi \):
Theorem 1 (Blackwell-Boyd Sufficiency Conditions) Let $T$ be a mapping from $(\Xi, \| \cdot \|)$ to $(\Xi, \| \cdot \|)$ such that (a) $T$ is monotone – if $f, g \in \Xi$ and $f \geq g$ implies $Tf \geq Tg$ – and (b) $T$ discounts constant functions – $\forall f \in \Xi$ and any constant function $A$, $T(f + A\phi) \leq Tf + \theta A\phi$ for some $\theta < 1$. Then $T$ is a strict contraction map.

Proof. Let $f, g \in \Xi$ and let $T$ satisfy (a) and (b). By the definition of $\| \cdot \|$, we must have

$$\frac{|f - g|}{\phi} \leq \|f - g\|$$

so that

$$f \leq g + \|f - g\| \phi.$$

By monotonicity,

$$Tf \leq T\left(g + \|f - g\| \phi\right).$$

By discounting constant functions,

$$Tf \leq Tg + \theta \|f - g\| \phi.$$

That is,

$$Tf - Tg \leq \theta \|f - g\| \phi.$$

Since the RHS is positive, we have

$$\frac{|Tf(x) - Tg(x)|}{\phi(x)} \leq \theta \|f - g\| \phi$$

\forall x. Therefore

$$\|Tf - Tg\| \phi \leq \theta \|f - g\| \phi.$$

That is, $T$ is a contraction. $\blacksquare$

To show that the Bellman operator is a strict contraction, we need to make assumptions on the primitives such that $T$ maps $(\Xi, \| \cdot \|)$ back to itself and is both monotone and discounting. These conditions turn out not to be particularly burdensome.

Theorem 2 Assume the one-period return function $r$ and the transition function $t$ are continuous on $X \times A$; the feasible actions correspondence $\Gamma$ is continuous and compact-valued; and there is a function $\phi \in \Xi$ with $\phi(x) > 0$ such that (i) there exists an $M$ with

$$\max_{a \in \Gamma(x)} \{r(x, a)\} \leq M\phi(x)$$

and (ii) there exists $\theta < 1$ such that

$$\beta \max_{a \in \Gamma(x)} \{\phi(t(x, a))\} \leq \theta \phi(x)$$

\forall x \in X. Under these conditions the Bellman operator is a strict contraction map.
Basically, this assumption says that the return function must be bounded in the current period by \( \phi \) and the state cannot evolve in such a way that it makes the \( \phi \) function grow too fast (which amounts to permitting the return to become unbounded at some point in the future). These are reasonable restrictions on any economy – a household should not be able to act in such a way as to get infinite current utility or infinite utility in the future. We now proceed to demonstrate that the Bellman equation, under the stated condition, will satisfy the sufficiency conditions.

It is easy to see that \( T : \Xi_\phi \rightarrow \Xi_\phi \); that is, if a function \( w \) is bounded by \( \phi \) then \( Tw \) will also be bounded by \( \phi \) – this follows from (i) of the condition. First, let \( v(x) \geq w(x) \) \( \forall x \in X \). Then

\[
(Tv)(x) = \max_{a \in \Gamma(x)} \{r(x, a) + \beta v(t(x, a))\} \\
\geq \max_{a \in \Gamma(x)} \{r(x, a) + \beta w(t(x, a))\} \\
= (Tw)(x).
\]

Monotonicity is proven. Now, let \( A \) be a constant and let \( w \in \Xi_\phi \). We have

\[
(T(w + A\phi))(x) = \max_{a \in \Gamma(x)} \{r(x, a) + \beta w(t(x, a)) + \beta A\phi(t(x, a))\} \\
\leq \max_{a \in \Gamma(x)} \{r(x, a) + \beta w(t(x, a))\} + \beta A \max_{a \in \Gamma(x)} \{\phi(t(x, a))\} \\
\leq (Tw)(x) + \theta A \phi(x).
\]

Discounting constant functions is proven if (ii) holds. We have shown that the Bellman operator is a contraction under the stated assumptions, which are almost entirely assumptions about the primitives of the model (except for the choice of \( \phi \), but we will turn to this in a moment).

We need only make one further step to show that there exist optimal paths that solve the sequence problem; after all, these are really the objects we are after. It may be the case that the value function is unattainable, that the max should be replaced by a sup. We are not interested in these types of problems – they do not solve the sequence problem we defined at the beginning of this section. To this end, note that the value function must satisfy

\[
v(x) \geq \sum_{t=0}^{\tau} \beta^t r(x_t, a_t) + \beta^\tau v(x_{\tau+1})
\]

for any \( \tau \geq 0 \) and the optimal sequence of actions. To match up with our previous problem in sequence form, we need only show that the "size" of the last term goes to zero as \( \tau \rightarrow \infty \). But we know that

\[
|v(x_{\tau+1})| = \frac{|v(x_{\tau+1})|}{\phi(x_{\tau+1})} \phi(x_{\tau+1}) \cdots \frac{\phi(x_1)}{\phi(x)} \phi(x)
\]

since \( \phi \) is always positive. Furthermore, we have

\[
\frac{|v(x_{\tau+1})|}{\phi(x_{\tau+1})} \leq \|v\|_{\phi} < \infty
\]

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because \( v \) is \( \phi \)-bounded. We also have, from our condition, that
\[
\frac{\phi(x_{s+1})}{\phi(x_s)} = \frac{\phi(t(x_s,a_s))}{\phi(x_s)} \leq \frac{\theta}{\beta}.
\]
Thus, we have
\[
|v(x_{\tau+1})| \leq \left(\frac{\theta}{\beta}\right)^\tau \|v\| \phi(x)
\]
or
\[
\beta^\tau |v(x_{\tau+1})| \leq \theta^\tau \|v\| \phi(x).
\]
The right hand side converges to zero because \( \theta < 1 \); therefore we have
\[
\beta^\tau |v(x_{\tau+1})| \to 0.
\]
We have thus proven that the value function exists; that it can be obtained by iterating on the Bellman operator; and that optimal paths generated by the policy function solve the sequence problem. This last property implies that once we are on an optimal path it is optimal to remain there, so that "plans" announced at time \( t \) for time \( t + k \) are carried out at \( t + k \). This is called the **Principle of Optimality**.

Returning to our example, we will use the one-sector growth model to demonstrate how to employ this machinery. Let \( u : \mathcal{R}_+ \to \mathcal{R} \) be increasing, continuous, and bounded below (note that this rules out the log function we used before but we won’t worry about that, that case can be handled by a different choice of \( \phi \)); since \( u \) is bounded below we can set \( u(0) = 0 \) without loss of generality. Let \( f : \mathcal{R}_+ \to \mathcal{R} \) be continuous and increasing with \( f(0) \geq 0 \). All of the continuity requirements for our condition are met. We need only find \( \phi \), the function which bounds the value function.

Since both \( u \) and \( f \) are bounded below by 0, let’s choose the function
\[
\phi(k) = 1 + u(f(k)).
\]
\( \phi \) is increasing, continuous, and strictly positive. (i) is satisfied since
\[
\max_{c \in \Gamma(k)} \{u(c)\} \leq M\phi(k)
\]
because \( c \leq f(k) \). Requirement (ii) will then be satisfied if
\[
\beta \sup_{k \geq 0} \left\{ \frac{\phi(f(k))}{\phi(k)} \right\} < 1.
\]
To check this condition we will need to put some structure on \( u \) and \( f \). For example, take the following functional forms:
\[
u(c) = \frac{c^\gamma}{\gamma}\]
for \( \gamma \in (0, 1) \) and
\[
f(k) = Ak
\]
with $A \geq 1$. We check the condition for (ii):

$$
\beta \sup_{k \geq 0} \left\{ \frac{1 + [(f \circ f)(k)]^{\gamma} / \gamma}{1 + [f(k)]^{\gamma} / \gamma} \right\} = \beta \sup_{k \geq 0} \left\{ \frac{1 + (A^2k)^{\gamma} / \gamma}{1 + (Ak)^{\gamma} / \gamma} \right\} = \beta A^\gamma.
$$

(To prove that the supremum is $\beta A^\gamma$ note that both the numerator and denominator are strictly increasing functions of $k$ and thus tend to $\infty$. Then use l'Hospital's rule.) Therefore we have the condition that optimal paths exist only if $\beta A^\gamma < 1$. Note that if $\gamma \leq 0$ $u$ is not bounded below, so as noted previously we need to choose a different $\phi$.

For some problems this machinery is more powerful than necessary. For example, if $u(f(k))$ is bounded both above and below – if either function is bounded above and below this condition will hold – then we can take $\phi(k) = 1$ for all $k$. Thus, all we need is $\beta < 1$ to ensure existence. We can also take $\phi$ to be constant if $f$ obeys the following ”maximum sustainable stock” condition:

**Definition 1** $f$ obeys the **maximum sustainable stock condition** if there exists $\bar{k} > 0$ such that $f(k) < k$ for all $k > \bar{k}$.

This condition in effect bounds the function $u(f(k))$ by placing a strict upper bound on the capital stocks we need consider: $k \in [0, L]$ where $L = \max\{k_0, \bar{k}\}$. If we do not start with capital above $\bar{k}$ we will never observe it, and if we do start with a capital stock that high we will be forced to run it down. The function $f(k) = Ak^\alpha$ is one which satisfies the maximum sustainable stock condition. Certain important classes of return functions – quadratic return functions – are always bounded above and below on any compact set; $\phi$ can be taken to be a constant in this case as well. When the production function is linear there may not exist a maximum capital stock, so unless the utility function is bounded (above and below) we require a nontrivial $\phi$ to prove existence.

One might think that dynamic programming puts severe restrictions on the type of functions that can be policy functions. It turns out that this is not the case – without more structure ”almost anything can happen,” a sort of counterpart to the Debreu-Mantel-Sonnenschein theorem.

**Theorem 3** (Boldrin-Montrucchio Theorem) Let $\Gamma(x) \subset \mathcal{R}$ be compact and convex for each $x \in X$ and let $\pi : X \to X$ be twice continuously differentiable. There exists $\beta \in (0, 1)$ and a return function $r(x, a)$ that is continuously-differentiable and concave such that

$$
\pi(x) = \arg\max_{a \in \Gamma(x)} \{r(x, a) + \beta v(t(x, a))\}
$$

where

$$
v(x) = \max_{a \in \Gamma(x)} \{r(x, a) + \beta v(t(x, a))\}.
$$
There is no economics in this theorem, so it may not really matter much, but it does put limits on what we can say without structure. In particular, the theorem says that really wild paths for state variables are consistent with them being generated by recursive decision problems, even so-called chaotic paths (paths which have cycles of infinite length). However, much like SMD, the conditions under which the really weird stuff happens are stringent; to get chaotic paths out of a dynamic programming problem we need $\beta$ to be very close to 0.

3 Properties of the Value Function

Now we have the value function. By the Theorem of the Maximum, we have that $v(k)$ is continuous under the assumptions given above. Also by the Theorem of the Maximum, the policy correspondence $a^* = \pi(k)$ is upper hemicontinuous and continuous if single-valued. What else can we prove about $v$ and $\pi$? It depends on the problem. As a concrete example, we will once again examine the optimal growth model. Let us assume that $u(c)$ and $f(k)$ are both increasing and concave. It is natural to assume that the value function will be increasing and concave as well; but is it?

First, assume that $k_t < \bar{k}_t$ are two initial states. Then we have

$$v(k_t) \geq \sum_{j=t}^{\infty} \beta^{j-t} u(c_j)$$

$$= u(c_t) + \sum_{j=t+1}^{\infty} \beta^{j-t} u(c_j).$$

Now assume that the agent has $\bar{k}_t$ instead of $k_t$. Since the agent can now consume an additional amount in period $t$, denoted $\bar{c}_t$, and still consume $c_j$ in every subsequent period, we must have

$$v(\bar{k}_t) \geq u(\bar{c}_t) + \sum_{j=t+1}^{\infty} \beta^{j-t} u(c_j).$$

By monotonicity of $u(c)$, we have

$$v(\bar{k}_t) > v(k_t).$$

Note that the agent may not choose to consume all this extra wealth; it simply is a feasible option.

Note first that if $w : X \rightarrow R$ is concave, then so is

$$(Tw)(k) = \max_{k'} \{u(f(k) - k') + \beta w(k')\}.$$  

To see this, take any $k, \bar{k} \geq 0$ and note that the set of feasible $k'$ is convex, given $k$. Then

$$(Tw)(\alpha k + (1 - \alpha)\bar{k}) \geq u(f(\alpha k + (1 - \alpha)\bar{k}) - \alpha k' - (1 - \alpha)\bar{k}') + \beta w(\alpha k' + (1 - \alpha)\bar{k}').$$
Using the fact that both $u$ and $w$ are concave, we have

$$u(f(\alpha k + (1 - \alpha)\kappa) - \alpha k' - (1 - \alpha)\kappa) + \beta w(\alpha k' + (1 - \alpha)\kappa) \geq \alpha(Tw)(k) + (1 - \alpha)(Tw)(\kappa).$$

That is, we know that $(Tw)(k)$ is concave; evidently, the Bellman operator takes concave functions into concave functions. Since the limit of any sequence of functions is the value function, all we need to prove is that the limit of a sequence of concave functions must be concave. But note that concave functions are defined by a weak inequality:

$$g(\alpha x + (1 - \alpha)y) \geq \alpha g(x) + (1 - \alpha)g(y).$$

Thus, the set defining it must be closed; that is, it must contain all its limit points. Therefore, the value function is concave. This proof will not establish that the value function is strictly concave since that is not a closed set.

What about differentiability – does $Dv(k)$ exist? The approach using the Bellman operator would fail – the limit of a sequence of differentiable functions may not be differentiable. Instead we approach the problem indirectly. Let $\pi$ denote the optimal decision rule for tomorrow’s capital. Now fix $k$ at some value $k^\ast$. If we can prove that the value function is differentiable here, we can claim differentiability everywhere on the interior of the constraint set. Consider the function

$$w(k) = u(f(k) - \pi(k^\ast)) + \beta v(\pi(k^\ast)).$$

Essentially, $w$ is the value of pretending one has $k^\ast$ capital today instead of $k$ but behaving optimally from this point onward. It should be clear that $w(k) \leq v(k)$ and that $w(k^\ast) = v(k^\ast)$. Since $u$ and $f$ are differentiable, $w$ is also differentiable with

$$Dw(k) = Du(f(k) - \pi(k^\ast))Df(k).$$

Note: we do not know if $\pi$ is differentiable. We can now state a useful lemma:

**Lemma 1 (Benveniste-Scheinkman Theorem)** Let $v$ be a real-valued, concave function defined on a convex set $D \subset \mathbb{R}^n$. If $w \in C^1$ is a concave function on a neighborhood $N$ of $x_0 \in D$ such that $w(x_0) = v(x_0)$ and $w(x) \leq v(x)$ $\forall x \in N$ then $v \in C^1$ at $x_0$ and $Dv(x_0) = Dw(x_0)$.

Essentially, this lemma states that if we can find a function that is everywhere below $v$, agrees with $v$ at $k^\ast$, and is continuously-differentiable at $k^\ast$, then $v$ will be continuously-differentiable at $k^\ast$ as well. $w(k)$ is that function, so the value function is differentiable and

$$Dv(k) = Du(f(k) - \pi(k^\ast))Df(k).$$

There are known conditions on the problem such that the value function is actually $C^2$, due to Santos (1991):
Proposition 1 Assume that $u$ is $C^2$ on the interior of the constraint set $D$ and one of the following holds:

1. $\alpha$-concavity: $\exists \alpha > 0$ such that
   
   $$ u(f(k) - k') + \frac{\alpha}{2} ||k'||^2 $$

   is concave for $k' = \pi(k)$ and $||D^2 u(f(k) - \pi(k))|| < \infty$;

2. $\forall (k, k') \in D$, $D^2 u$ is negative definite and $\forall k' = \pi(k)$,

   $$ \left[ ||D^2 u(f(k) - \pi(k'))|| \right] \cdot \left[ ||D^2 u(f(k) - \pi(k))^{-1}|| \right] < \infty. $$

Then $v$ is $C^2$.

These conditions amount to making sure the utility function plus a particular convex function is still concave, plus an assumption that the second derivative does not vanish or explode.

We may want to rule out boundary solutions in our models – for example, in an economy with only Robinson Crusoe it will not make much sense if we examine solutions where all the existing capital is consumed. We therefore examine conditions – known as Inada conditions – which guarantee an interior solution to our problem. First, let the utility function be such that $u'(c) \rightarrow \infty$ as $c \rightarrow 0$. We further assume that the utility function obeys $u'(c) \rightarrow 0$ as $c \rightarrow \infty$; that is, marginal utility vanishes for large enough consumption. We also assume the production function obeys the same conditions for all of its marginal products. It is easy to see that consumption will never be zero – the marginal utility would be infinite and thus an increase of $\epsilon$ would definitely increase utility for any finite cost of consumption. Similarly, next period’s capital will also never be zero – the return on capital $f'(k)$ would exceed any cost. We therefore have interior decisions for both consumption and savings. The upper boundary conditions ensure that both remain bounded by driving the return below any positive cost.

4 Euler Equations

One of the problems with dynamic programming is that the form of the value function will typically be unknown. We can get around this problem using tools from variational calculus – the optimality conditions for such problems are known as Euler equations. Euler equations are useful for analyzing the dynamics of our system and often are easier to use for intuitive purposes.

Let’s take the Bellman equation for the growth model:

$$ v(k) = \max_{k'} \{ u(f(k) - k') + \beta v(k') \}. $$
Assuming enough structure for interior differentiable solutions, the first-order condition is
\[ u'(f(k) - k') = \beta v'(k'). \]

This does not seem like much of an improvement; while we don’t need to know the value function we need to know its derivative. Let’s assume that the above equation is satisfied by the policy function
\[ k' = \pi(k). \]

Inserting this into the Bellman equation gives us
\[ v(k) = u(f(k) - \pi(k)) + \beta v(\pi(k)). \]

Since this must hold at every feasible \( k \), we can differentiate (at least if we have an interior optimum), yielding
\[ v'(k) = u'(f(k) - \pi(k))f'(k) - [u'(f(k) - \pi(k)) - \beta v'(\pi(k))]\pi'(k). \]

Collecting terms involving \( \pi'(k) \) we have
\[ v'(k) = u'(f(k) - \pi(k))f'(k) - \beta v'(\pi(k))\pi'(k). \]

From the first-order condition the term in the square brackets is zero. This leaves us with an envelope theorem result that
\[ v'(k) = u'(f(k) - \pi(k))f'(k). \]

This envelope condition must also hold at every \( k \); in particular, it holds at \( k' = \pi(k) \). This implies
\[ v'(k') = u'(f(k') - \pi(k'))f'(k'). \]

Inserting this term into the first-order condition we reach the Euler equation
\[ u'(f(k) - \pi(k)) = \beta u'(f(k') - \pi(k'))f'(k') \]

or
\[ u'(f(k) - \pi(k)) = \beta u'(f(\pi(k)) - \pi(\pi(k)))f'(\pi(k)). \]

One thing that is important is that we could obtain the same result by the following method (think of the above demonstration as another proof of the envelope theorem) for this problem. Let the Bellman equation be
\[ v(k) = u(f(k) - \pi(k)) + \beta v(\pi(k)). \]

Differentiate both sides with respect to \( k \) but ignore the dependence of \( \pi(k) \) on \( k \); that is, pretend that the equation is actually
\[ v(k) = u(f(k) - k') + \beta v(k'). \]
We thus obtain
\[ v'(k) = u'(f(k) - \pi(k))f'(k) \]
which is exactly the envelope condition we obtained above.

Note that we can use this fact to prove \( v(k) \) is increasing. The proof is very simple:
\[ v'(k) = u'(f(k) - \pi(k))(f'(k) - \pi'(k)) + \beta v'(\pi(k))\pi'(k) \]
\[ = u'(f(k) - \pi(k))f'(k) - \pi'(k)(u'(f(k) - \pi(k)) - \beta v'(\pi(k))) \]
\[ = u'(f(k) - \pi(k))f'(k) > 0 \]
since both \( u \) and \( f \) are increasing and the term multiplying \( \pi'(k) \) is zero from the foc.

Similarly, we can differentiate to prove that the value function is concave near the steady state, provided that \( \pi'(k) \) actually exists and is bounded in absolute value:
\[ v''(k) = u''(f(k) - \pi(k))(f'(k) - \pi'(k))^2 + u'(f(k) - \pi(k))(f''(k) - \pi''(k)) + \beta v''(\pi(k))(\pi'(k))^2 \]
\[ = u''(f(k) - \pi(k))(f'(k) - \pi'(k))^2 + \beta v''(\pi(k))(\pi'(k))^2 + u'(f(k) - \pi(k))f''(k). \]
Setting \( \pi(k) = k \) we can solve for \( v''(k) \):
\[ v''(k) = \frac{u''(f(k) - \pi(k))(f'(k) - \pi'(k))^2 + u'(f(k) - \pi(k))f''(k)}{1 - \beta \pi'(k)^2} < 0 \]
under the condition that \( |\pi'(k)| < 1 \). Important: we cannot simply differentiate the envelope expression to get \( v'' \), because the envelope condition is not an identity. Of course, we have already proven that the value function is concave, so this approach can be viewed as putting bounds on the derivative of the policy function. Most importantly, this proves that the steady state is stable, so that the sequence
\[ \{k, g(k), g(g(k)), \ldots\} \]
converges to the steady state.

We can now explore how \( k' = \pi(k) \) and \( c = f(k) - \pi(k) \) depend on \( k \). It seems natural that both would be increasing in \( k_t \), but can we prove it? Assume as before that \( k_t < \bar{k}_t \); can we conclude that
\[ \pi(k_t) < \pi(\bar{k}_t) \]
and
\[ f(k_t) - \pi(k_t) < f(\bar{k}_t) - \pi(\bar{k}_t)? \]
Under the assumption that \( \pi \) is differentiable (and even if it is not, we can do this in finite differences) the first-order condition for optimality
\[ u'(f(k) - \pi(k)) = \beta v'(\pi(k)) \]
can be differentiated to obtain
\[ \pi'(k) = \frac{u''(f(k) - \pi(k))f'(k)}{u''(f(k) - \pi(k)) + \beta v''(\pi(k))}. \]
Since $u''$ and $v''$ are negative, this expression is positive; furthermore, it is bounded in absolute value by 1, as discussed above. Therefore, savings is increasing in capital. The same steps imply that

$$f'(k) - \pi'(k) = \frac{\beta v''(\pi(k)) \pi'(k)}{u''(f'(k) - \pi(k))} > 0,$$

so that consumption is also increasing in capital.

We could solve the Euler equation for the policy function $\pi(k)$ (at least in principle) if we were certain that it constituted a necessary and sufficient condition for the problem at hand. Unfortunately, it does not – the problem being that we have discarded information about the value function. To more clearly see this, let’s write this in terms of sequences:

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}).$$

This is a second-order difference equation in the capital stock. As we have seen before, second-order difference equations require two boundary conditions, and we have only one: the initial capital stock $k_0$. At time 1, the Euler equation is

$$u'(f(k_0) - k_1) = \beta u'(f(k_1) - k_2) f'(k_1).$$

If we somehow knew $k_1$ we could use the above equation to solve for $k_2$ and move forward. But we are supposed to learn what $k_1$ is from the model; this is not an appropriate condition to impose. We will therefore look for a “terminal condition” on the capital stock – something that happens at the end of the time horizon that will pin down the path $\{k_t\}$ uniquely. Remembering back to our finite horizon case, one of the necessary conditions for optimality was

$$k_{T+1} = 0.$$

That is, the household set the capital stock to zero in the last period of life. Something analogous will hold here. Suppose that $u$ and $f$ are both concave – we have already shown that this is sufficient for $v$ to be concave as well. Concave functions satisfy the following inequality:

$$g(z^*) + g'(z^*)(z - z^*) \geq g(z)$$

for every set of points $(z, z^*)$. If we can assume that $u$ and $f$ are bounded below we know that the value function will also be bounded below – we can therefore freely set $v(0) = 0$. With concavity we have

$$v(k_{t+1}) + v'(k_{t+1})(0 - k_{t+1}) \geq v(0)$$

or

$$v(k_{t+1}) \geq v'(k_{t+1}) k_{t+1} \geq 0.$$

(The last inequality comes from the envelope condition.) Since we know that $v'(k) = f'(k) u'(f(k) - k')$ we have

$$\beta^{t+1} v(k_{t+1}) \geq \beta^{t+1} k_{t+1} f'(k_{t+1}) u'(f(k_{t+1}) - k_{t+2}) \geq 0.$$
Furthermore, along an optimal path we have already shown that
\[ \beta^{t+1} v(k_{t+1}) \to 0. \]

Therefore, we establish that it is necessary for an optimal path to satisfy
\[ \beta^{t+1} k_{t+1} f'(k_{t+1}) u'(f(k_{t+1}) - k_{t+2}) \to 0. \]

Making one final substitution from the Euler equation we get
\[ \beta^{t+1} k_{t+1} u'(f(k_t) - k_{t+1}) \to 0. \]

This is the transversality condition – it is the limit of analogous finite horizon results that require capital to be nonnegative in the final period of life.

What is the interpretation of the Euler equation in the growth model? The term \( u'(c_t) \) is the cost of reducing current consumption by \( \epsilon \) and saving it instead. The term \( \beta u'(c_{t+1}) f'(k_{t+1}) \) is the benefit in consumption tomorrow if the consumer eats the extra income from the additional \( \epsilon \) of saving. If we are on an optimal interior path this rearrangement of consumption cannot change utility because it is feasible; similarly, we can do the opposite and increase current consumption by \( \epsilon \), reversing the interpretation of the two terms. Thus, if the current path is optimal, no such deviation can increase utility, so the cost and benefit must be equal (the Euler equation must hold).

Since we can rewrite the Euler equation in terms of multiple periods deviations of more than 1 period cannot be optimal either:

\[
\begin{align*}
    u'(c_t) &= \beta u'(c_{t+1}) f'(k_{t+1}) \\
    &= \beta [\beta u'(c_{t+2}) f'(k_{t+2})] f'(k_{t+1}) \\
    &= \beta^2 u'(c_{t+2}) f'(k_{t+2}) f'(k_{t+1}).
\end{align*}
\]

The transversality condition rules out a permanent deviation – the value of the extra capital generated by the increase in saving (measured in marginal utility terms) shrinks sufficiently quickly.

A loose end: we have proven that the Euler equations and the TVC are necessary conditions (that is, they are implied by the policy function associated with the solution to the Bellman equation), but not that they are sufficient (that is, solving them generates the value function that solves the Bellman equation). Suppose \( \{k_t\}_{t=0}^{\infty} \) is a path from \( k_0 \) that satisfies both, with associated path \( \{c_t\}_{t=0}^{\infty} \). Since \( u \) is concave, we have
\[
u(c_t) + Du(c_t)(c - c_t) \geq u(c) \]
\( \forall c \geq 0 \). Rearranging, we obtain
\[
    u(c_t) - u(c) \geq Du(c_t)(c_t - c). 
\]
Thus, for any feasible path \( \{ \tilde{c}_t \}_{t=0}^{\infty} \) from \( k_0 \) we have

\[
\sum_{t=0}^{T} \beta^t (u(c_t) - u(\tilde{c}_t)) \geq \sum_{t=0}^{T} \beta^t Du(c_t) (c_t - \tilde{c}_t)
\]

\( \forall T \). Also,

\[
c_t - \tilde{c}_t = f(k_t) - k_{t+1} - \left( f\left(\tilde{k}_t\right) - \tilde{k}_{t+1}\right).
\]

Since \( f \) is concave, we have

\[
f(k_t) - f\left(\tilde{k}_t\right) \geq f'(k_t) (k_t - \tilde{k}_t).
\]

Therefore,

\[
c_t - \tilde{c}_t \geq f'(k_t) (k_t - \tilde{k}_t) - \left( k_{t+1} - \tilde{k}_{t+1}\right).
\]

That is, we have

\[
\sum_{t=0}^{T} \beta^t Du(c_t) (c_t - \tilde{c}_t) \geq \sum_{t=0}^{T} \beta^t Du(c_t) \left( f'(k_t) \left( k_t - \tilde{k}_t\right) - \left( k_{t+1} - \tilde{k}_{t+1}\right)\right).
\]

The first term of the RHS is

\[
Du(c_0) \left( Df(k_0) (k_0 - k_0) - (k_1 - \tilde{k}_1)\right) = -Du(c_0) \left( k_1 - \tilde{k}_1\right).
\]

The second term is

\[
Du(c_1) \left( Df(k_1) \left( k_1 - \tilde{k}_1\right) - (k_2 - \tilde{k}_2)\right) = \beta Du(c_1) f'(k_1) \left( k_1 - \tilde{k}_1\right) - \beta Du(c_1) \left( k_2 - \tilde{k}_2\right).
\]

Adding the two together yields

\[
\left( \beta Du(c_1) Df(k_1) - Du(c_0) \right) \left( k_1 - \tilde{k}_1\right) - \beta Du(c_1) \left( k_2 - \tilde{k}_2\right)
\]

but the first term is zero by the Euler equation, so we’re left with only

\[
-\beta Du(c_1) \left( k_2 - \tilde{k}_2\right).
\]

Adding the third term to the first two leaves us with only

\[
-\beta^2 Du(c_2) \left( k_3 - \tilde{k}_3\right)
\]

and so on. For arbitrary \( T \), we find that the sum becomes

\[
-\beta^T Du(c_T) \left( k_{T+1} - \tilde{k}_{T+1}\right).
\]
Therefore, we have
\[
\sum_{t=0}^{T} \beta^t (u(c_t) - u(\tilde{c}_t)) \geq -\beta^T Du(c_T) \left(k_{T+1} - \tilde{k}_{T+1}\right)
\]
\[
\geq -\beta^T Du(c_T) k_{T+1}
\]
because \(\beta^T Du(c_{T+1}) \tilde{k}_{T+1} \geq 0\). Now we employ the transversality condition, which says that the RHS of the equation above converges to 0. Thus,
\[
\sum_{t=0}^{\infty} \beta^t (u(c_t) - u(\tilde{c}_t)) \geq 0
\]
or
\[
\sum_{t=0}^{\infty} \beta^t u(c_t) \geq \sum_{t=0}^{\infty} \beta^t u(\tilde{c}_t).
\]
Since the path \(\{\tilde{c}_t\}_{t=0}^{\infty}\) was arbitrary, \(\{c_t\}_{t=0}^{\infty}\) must be optimal. If we impose strict concavity on \(u\) we get the inequality is also strict, so that \(\{c_t\}_{t=0}^{\infty}\) is strictly better than any alternative.

5 Properties of the Value Function using the Sequential Problem Only

Let \(\{c_0, c_1, ...\}\) denote a consumption path and \(U(c_0, c_1, ...\) denote the utility of that path. Further assume that
\[
U(C) = \sum_{t=0}^{\infty} \beta^t u(c_t)
\]
with \(0 < \beta < 1\) and \(u\) is twice continuously differentiable and satisfies \(u(0) = 0\), \(Du > 0\), \(D^2u < 0\), and \(Du(0) = \infty\). Define the value function by
\[
V(s) = \sup_{\{c_0, c_1, ...\}} \sum_{t=0}^{\infty} \beta^t u(c_t) < \infty
\]
subject to
\[
c_0 + x_0 = s
\]
\[
c_t + x_t = f(x_{t-1})
\]
\[
c_t > 0
\]
\[
x_t > 0
\]
where \(f\) has the same properties as \(u\) along with \(Df(\infty) = 0\).

Lemma 2 \(V(s)\) is increasing and strictly concave.
Proof. Consider \( s_1 < s_2 \). Then

\[
V(s_1) = \sup_{\{x_0,c_1,\ldots\}} u(s_1 - x_0) + \sum_{t=1}^{\infty} \beta^t u(c_t)
\]

and

\[
V(s_2) = \sup_{\{x_0,c_1,\ldots\}} u(s_2 - x_0) + \sum_{t=1}^{\infty} \beta^t u(c_t)
\]

for some feasible sequences of future consumption. It is feasible to choose the same \( x_0 \) for both problems and therefore the same sequence of future consumptions, so

\[
V(s_2) - V(s_1) = u(s_2 - x_0) - u(s_1 - x_0) > 0
\]

because \( u \) is increasing.

Consider \( s_1 > 0, s_2 > 0, \) and \( 0 < \lambda < 1 \). To each \( \epsilon > 0 \) there exists feasible consumption plans \( C^1 = \{c_1^1\} \) and \( C^2 = \{c_1^2\} \) from \( s_1 \) and \( s_2 \) such that

\[
V(s_1) \leq \sum_{t=0}^{\infty} \beta^t u(c_1^1) + \epsilon
\]

and

\[
V(s_2) \leq \sum_{t=0}^{\infty} \beta^t u(c_1^2) + \epsilon.
\]

The path \( C = \{\lambda c_1^1 + (1 - \lambda) c_1^2\} \) is feasible from \( \lambda s_1 + (1 - \lambda) s_2 \) because \( f \) is concave. Therefore

\[
V(\lambda s_1 + (1 - \lambda) s_2) \geq \sum_{t=0}^{\infty} \beta^t \left( \lambda u(c_1^1) + (1 - \lambda) u(c_1^2) \right)
\]

\[
> \sum_{t=0}^{\infty} \beta^t \left( \lambda V(c_1^1) + (1 - \lambda) V(c_1^2) \right)
\]

\[
\geq \lambda V(s_1) + (1 - \lambda) V(s_2) - \epsilon.
\]

Therefore \( V \) is strictly concave since \( \epsilon \) was arbitrary.

**Corollary 1** \( V(s) \) is continuous \( \forall s > 0 \).

Proof. Continuity of \( V(s) \) follows from the concavity of \( V(s) \).

Therefore

\[
V(s) = \max_{0 \leq c \leq s} \{u(c) + \beta V(f(s - c))\}.
\]

The maximum is achieved because the maximand is a continuous function of \( c \) and \([0, s]\) and is not achieved at \( c = 0 \) since \( Du(0) = \infty \). The maximum is achieved at a unique \( c \) because the maximand is a strictly concave function. Therefore denote the unique optimal policy by \( c = g(s) \).
Lemma 3  \( DV(s) \) exists \( \forall s > 0 \) and \( DV(s) = Du(g(s)) \) \( \forall s > 0 \).

**Proof.** Let \( \xi > 0 \) and \( \overline{C} \) be an optimal consumption plan from \( \xi \). Define a feasible consumption plan from \( \xi + \Delta \xi \) where \( \Delta \xi > 0 \) as follows:

\[
\begin{align*}
c_0 &= \overline{v}_0 + \Delta \xi \\
c_t &= \overline{v}_t \forall t \geq 1.
\end{align*}
\]

Then

\[
\begin{align*}
V(\xi + \Delta \xi) - V(\xi) &\geq \sum_{t=0}^{\infty} \beta^t (u(c_t) - u(\overline{v}_t)) \\
&= u(\overline{v}_0 + \Delta \xi) - u(\overline{v}_0) \\
&= Du(\overline{v}_0) \Delta \xi + O(\Delta \xi).
\end{align*}
\]

Therefore,

\[
DV_+ (\xi) \geq Du(\overline{v}_0).
\]

Let \( 0 < \Delta \xi < \overline{v}_0 \). Define a feasible consumption plan from \( \xi - \Delta \xi \) as follows:

\[
\begin{align*}
c_0 &= \overline{v}_0 - \Delta \xi \\
c_t &= \overline{v}_t \forall t \geq 1.
\end{align*}
\]

Then

\[
\begin{align*}
V(\xi - \Delta \xi) - V(\xi) &\geq \sum_{t=0}^{\infty} \beta^t (u(c_t) - u(\overline{v}_t)) \\
&= u(\overline{v}_0 - \Delta \xi) - u(\overline{v}_0) \\
&= -Du(\overline{v}_0) \Delta \xi + O(\Delta \xi).
\end{align*}
\]

Therefore,

\[
DV_- (\xi) \leq Du(\overline{v}_0).
\]

Thus, \( DV(\xi) = Du(\overline{v}_0) \). But \( \overline{v}_0 = g(\xi) \), so \( DV(\xi) = Du(g(\xi)) \). □

Using the Bellman equation we therefore get

\[
\beta DV(f(s-c)) Df(s-c) = Du(c).
\]

Lemma 4 Let \( g(x) \) be the optimal policy function and let \( (\overline{X}, \overline{C}) \) be the optimal program defined by \( g(x) \). Then

\[
Du(\overline{v}_t) = \beta Df(\overline{v}_t) Du(\overline{v}_{t+1}).
\]
Proof. Let $s > 0$. Then it follows that

$$Du(g(s)) = \beta DV(f(s - g(s))) Df(s - g(s))$$

and so by the previous result

$$Du(g(s)) = \beta Du(g(f(s - g(s)))) Df(s - g(s)).$$

Setting $s = f(x_{t-1})$ implies

$$Du(\tau_t) = \beta Du(\tau_{t+1}) Df(\pi_t).$$

$\blacksquare$