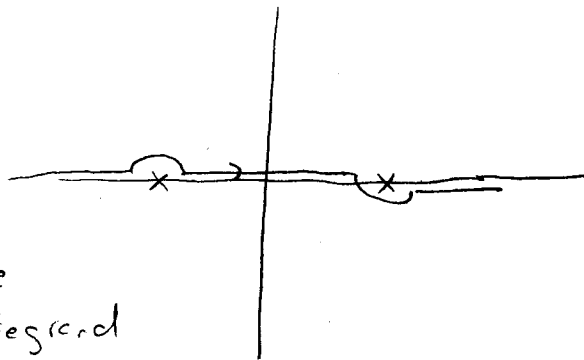


## Supplement 7 - Alternative Green's function

To calculate the Green's function for scattering, we had to deform the contour in our integral.

We used



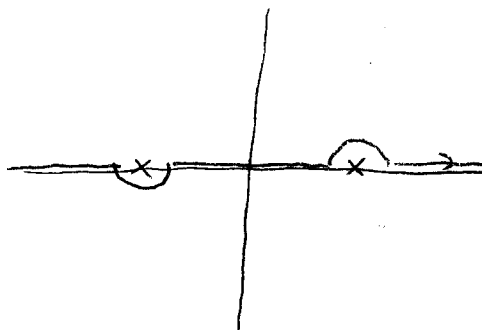
where x's indicate the poles of the integrand

I said that deforming the contour differently would give a different Green's function  $G'$  that still solved

$$(\nabla^2 + k^2)G' = \delta^3(r)$$

I'll demonstrate that here.

Say we used contour:



For  $\int_{-\infty}^{\infty} \frac{s e^{isr}}{k^2 s^2} ds$ , close in upper half plane:

$$= - \oint \left( \frac{s e^{isr}}{s-k} \right) \frac{1}{s+k} ds$$

$$= -2\pi i \frac{(-k) e^{-ikr}}{(-2k)} = -\pi i e^{-ikr}$$

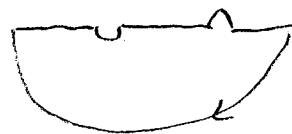


For  $\int_{-\infty}^{\infty} \frac{se^{-sr}}{k^2-s^2} ds$ , close in lower half plane

$$= -\oint \left( \frac{se^{-sr}}{s+k} \right) \frac{1}{s-k} ds$$

$$= +2\pi i \frac{ke^{-kr}}{2k}$$

$$= \pi i e^{-kr}$$



(including minus sign because contour is traverse in clockwise sense)

$$\text{So } G(r) = \frac{1}{4\pi^2} \frac{1}{2ir} \left[ \int_{-\infty}^{\infty} \frac{se^{sr}}{k^2-s^2} ds - \int_{-\infty}^{\infty} \frac{se^{-sr}}{k^2-s^2} ds \right]$$

$$= \frac{1}{4\pi^2} \frac{1}{2ir} \left[ -\pi i e^{-kr} - \pi i e^{-kr} \right]$$

$$= -\frac{1}{4\pi} \frac{e^{-kr}}{r}$$

With other contour, got  $G(r) = -\frac{1}{4\pi} \frac{e^{+kr}}{r}$

I claim that  $(\nabla^2 + k^2)(G - G') = 0$

$$\text{Note } G - G' = -\frac{2i}{4\pi} \frac{\sin kr}{r} \equiv H(r)$$

$$\nabla^2 H = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} H = -\frac{2i}{4\pi r^2} \frac{\partial}{\partial r} r^2 \left( \frac{\cos kr}{r} - \frac{\sin kr}{r^2} \right)$$

$$= -\frac{2i}{4\pi r^2} \frac{\partial}{\partial r} (kr \cos kr - \sin kr)$$

$$= -\frac{2i}{4\pi r^2} (k \cos kr - k^2 r \sin kr - k \cos kr)$$

$$= \frac{2ik^2}{4\pi} \frac{\sin kr}{r} = -k^2 H$$

Thus  $(\nabla^2 + k^2)H = 0$  as claimed.